Ch 2

$$\boxed{\text{METHOD 5 : EXPEND AND CONTRACT}}$$

Method: replace the original single sum by seemingly more complicated sum (DOUBLE sum) that can be, in turn simplified.

EXAMPLE (of replacement)

$$\boxed{\sum_{K=1}^{n} K^2 = \sum_{1 \le j \le K \le n} K}$$

$$K = \sum_{j=1}^{K} 1$$

Proof:

$$\boxed{\sum_{K=1}^{n} K^2} = \sum_{K=1}^{n} K \cdot K = \sum_{K=1}^{n} K \left(\sum_{j=1}^{K} 1 \right) =$$

$$= \sum_{K=1}^{n} \sum_{j=1}^{K} K = \sum_{j=1}^{K} \sum_{K=1}^{n} K = \boxed{\sum_{1 \le j \le K \le n} K}$$

$1 \le j \le K \wedge 1 \le K \le n \equiv 1 \le j \le K \le n$

$\sum_{i=1}^{K}$ when $j > K$ DNE Does not exist

Problem

M5(1a)

Show that in this case

$$\boxed{\begin{gathered} 1 \le j \le k \cap 1 \le k \le n \\ \equiv 1 \le j \le k \le n \end{gathered}}$$

i.e

$\boxed{\text{when } j \le k}$

Become

$$\sum_{j=1}^{k} \sum_{k=1}^{n} a_{ij}$$

when $j > k$

first sum $\sum_{j=1}^{k}$ DNE

Does not exist

We can hence replace

Property

$$\boxed{\sum_{j=1}^{k} \sum_{k=1}^{n} a_{ij} = \sum_{1 \le j \le k \le n} a_{ij}.}$$

EXAMPLE (of Method 5) M5(2)

$$\square_n = \boxed{\sum_{k=1}^{n} k^2} = \sum_{1 \le j \le k \le n} k =$$

$$= \sum_{j=1}^{n} \left(\sum_{k=j}^{n} k \right) = \sum_{j=1}^{n} \frac{(j+n)}{2}(n-j+1)$$

$$= \frac{1}{2} \sum_{j=1}^{n} \left(n(n+1) + j - j^2 \right)$$

$$= \frac{1}{2} n^2(n+1) + \frac{1}{2} \sum_{j=1}^{n} j - \frac{1}{2} \sum_{j=1}^{n} j^2$$

$$= \frac{1}{2} n^2(n+1) + \frac{1}{4} n(n+1) - \frac{1}{2} \boxed{\sum_{k=1}^{n} k^2}$$

$$= \frac{1}{2} n\left(n+\frac{1}{2}\right)(n+1) - \frac{1}{2} \square_n$$

$$\frac{3}{2} \square_n = \frac{1}{2} n\left(n+\frac{1}{2}\right)(n+1)$$

$$\boxed{\square_n = \frac{n(n+1/2)(n+1)}{3}}$$

$$\sum_{j=1}^{n} n(n+1) = n(n+1) \sum_{j=1}^{n} 1 = n^2(n+1)$$

$\boxed{\text{EXAMPLE}}$ (of replacement)

Prove that

$$\boxed{\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 = 2\cdot\sum_{1\le j\le k\le n} j\cdot k}$$

to use this property in ch2 prob 15 Hmwk.

Proof.

STEP 1. Evaluate $\sum_{k=1}^{n} k^3 + (n+1)^3$

in terms $\sum_{k=1}^{n} k^2$ AND $\sum_{1\le j\le k\le n} k\cdot j$

$$\sum_{k=1}^{n} k^3 + (n+1)^3 = \sum_{k=0}^{n-1} (k+1)^3 + (n+1)^3$$

Perturb the sum

$$= \sum_{k=0}^{n} (k+1)^3 = \sum_{k=1}^{n} (k+1)^3 + 1$$

$$(k+1)^3 = (k^2+2k+1)\cdot(k+1) = k^2(k+1) + (2k+1)(k+1)$$

$$= 1 + \sum_{k=1}^{n} \underbrace{k^2(k+1)}_{k\cdot k(k+1)} + \sum_{k=1}^{n} (2k+1)(k+1)$$

+ use $\boxed{k(k+1) = 2\sum_{j=1}^{k} j}$

$$= 1 + \sum_{k=1}^{n} k\left(2\sum_{j=1}^{k} j\right) + \sum_{k=1}^{n} (2k^2+3k+1)$$

M5(4)

$$\boxed{\sum_{k=1}^{n} k^3 + (n+1)^3} = 1 + 2\sum_{k=1}^{n}\sum_{j=1}^{k} k\cdot j +$$

proved

$$+ 2\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$\frac{3}{2} n(n+1)$

$$= \boxed{1 + 2\sum_{1\le j\le k\le n} kj + \frac{3}{2}n(n+1) + n + 2\sum_{k=1}^{n} k^2}$$

Add to both sides

$\sum_{k=1}^{n} k^2 - (n+1)^3$ we get

$$\boxed{\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2} = 2\sum_{1\le j\le k\le n} kj + \underbrace{3\sum_{k=1}^{n} k^2}_{= n(n+1/2)(n+1)} + 3\cdot\frac{n(n+1)}{2}$$

$$+ (n+1) - (n+1)^3$$

$$= \boxed{2\sum_{1\le j\le k\le n} kj} + n(n+1/2)(n+1) + \frac{3}{2}n(n+1) + (n+1) - (n+1)^3 = 0$$

evaluate this part!

Evaluation

$$n(n+\tfrac{1}{2})(n+1) + \tfrac{3}{2}n(n+1) + (n+1) - (n+1)^3 =$$

$$= (n+1)\left(n(n+\tfrac{1}{2}) + \tfrac{3}{2}n + 1 - (n+1)^2\right)$$

$$= (n+1)\left(\cancel{n^2} + \underbrace{\tfrac{1}{2}n + \tfrac{3}{2}n} + \cancel{1} - \cancel{n^2} - 2n - \cancel{1}\right)$$

$$(n+1)(2n - 2n) = 0$$

We proved:

$$\boxed{\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 = 2\sum_{1\le j\le k\le n} k\cdot j}$$

We used

$$2\sum_{k=1}^{n}\sum_{j=1}^{k} k\cdot j = 2\sum_{j=1}^{k}\sum_{k=1}^{n} k\cdot j$$

proved already

$$= 2\sum_{1\le j\le k\le n} kj$$

ANOTHER PROOF

$$\boxed{\sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k^3 = 2 \sum_{1 \le j \le k \le n} k}$$

$$\boxed{\sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k^3} = 0^2 + 0^2 + \sum_{k=1}^{n} \left(k^2 + k^3\right)$$

use

$$\boxed{\frac{k(k+1)}{2} = \sum_{j=1}^{k} j}$$

$$= \sum_{k=1}^{n} k^2 (k+1)$$

$$= \sum_{k=1}^{n} 2k \cdot \frac{k(k+1)}{2}$$

$$= 2 \sum_{k=1}^{n} k \cdot \sum_{j=1}^{k} j$$

cons on j

$$= 2 \sum_{k=1}^{n} \sum_{j=1}^{k} k j$$

use

$$\boxed{\sum_{k=1}^{n} \sum_{j=1}^{k} a_{jk} = \sum_{1 \le j \le k \le n} a_{jk}}$$

$$= \boxed{2 \sum_{1 \le j \le k \le n} j \cdot k}$$

end.