SHORT REVIEW PROBLEMS

Chapter 1 & Chapter 2

Infinite Sums
(end of Chapter 2)
on THURSDAY
I will put
LECTURE NOTES
and Slides on
the Web
+ Home-work Problems
on Infinite Sums.

SOME SOLUTIONS

Q1 Evaluate

$$\boxed{S_n = \sum_{k=0}^{n} (-1)^k}$$

Develop a general formula.

$$S_n + (-1)^{n+1} = 1 + \sum_{k=0}^{n} (-1)^{k+1}$$

$$S_n - (-1)^n = 1 + \sum_{k=0}^{n} (-1)^k(-1)$$

$$S_n - (-1)^n = 1 - S_n$$

$(-1)^{n+1} = -(-1)^n$

$$2S_n = 1 + (-1)^n$$

$$\boxed{S_n = \frac{1 + (-1)^n}{2}}$$

$$S_n = \begin{cases} 1 & n \in \text{Even} \\ 0 & n \in \text{Odd} \end{cases}$$

Check

$$S_3 = (-1)^0 + (-1) + (-1)^2 + (-1)^3$$
$$= 1 - 1 + 1 - 1 = 0$$

$$S_3 = \frac{1+(-1)^3}{2} = 0 \quad \text{yes.}$$

Q3 Evaluate

$$\boxed{S_n = \sum_{k=0}^{n} (-1)^k k}$$

$a_n = (-1)^k \cdot k$

$a_{n+1} = (-1)^{n+1}(n+1)$

$$S_n + (-1)^{n+1}(n+1) = \sum_{k=0}^{n} (-1)^{k+1}(k+1)$$

$$S_n - (-1)^n(n+1) = -\sum_{k=0}^{n}(-1)^k(k+1) =$$

$$= -\overbrace{\sum_{k=0}^{n}(-1)^k k}^{S_n} - \sum_{k=0}^{n}(-1)^k$$

$$2S_n = (-1)^n(n+1) - \sum_{k=0}^{n}(-1)^k$$

$$2S_n = (-1)^n(n+1) - \frac{1+(-1)^n}{2}$$

$$\boxed{S_n = \frac{1}{2}\left((-1)^n(n+1) - \frac{1+(-1)^n}{2}\right)}$$

Q3 Evaluate

$$\boxed{S_n = \sum_{k=0}^{n} (-1)^k k^2}$$

$$S_n + (-1)^{n+1}(n+1)^2 = \sum_{k=0}^{n} (-1)^{k+1}(k+1)^2$$

$$S_n - (-1)^n(n+1)^2 = -\sum_{k=0}^{n} (-1)^k (k^2 + 2k + 1)$$

$$= -\left(\sum_{k=0}^{n} (-1)^k k^2 + 2\sum_{k=0}^{n} (-1)^k k + \sum_{k=0}^{n} (-1)^k\right)$$

$$= -S_n - \underbrace{2\sum_{k=0}^{n} (-1)^k k}_{} - \sum_{k=0}^{n} (-1)^k$$

$$2S_n = (-1)^n(n+1)^2 - \left((-1)^n(n+1) - \frac{1+(-1)^n}{2}\right)$$

$$- \left(\frac{1+(-1)^n}{2}\right)$$

common factor

$$2S_n = (-1)^n(n+1)^2 - (-1)^n(n+1) = (-1)^n(n+1)(n+1-1)$$

$$\boxed{S_n = \frac{(-1)^n(n+1)\,n}{2}}$$

QUESTION 4

Use REPERTOIRE method

$$\boxed{S_n = \sum_{k=0}^{n} k^2}$$

Recursive formula

$$\boxed{\begin{gathered} S_n = S_{n-1} + n^2 \\ S_0 = 0 \end{gathered}}$$

is a special case of

$$\boxed{\begin{aligned} S_0 &= \alpha \\ S_n &= S_{n-1} + \beta n^2 + \gamma n + \delta \end{aligned}}$$

$(\alpha=0,\ \beta=1,\ \gamma=0,\ \delta=$

closed formula

$$\boxed{S(n) = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta}$$

CASE 1

① Let $\boxed{S_n = 1}$ for all $n \in \mathbb{N}$

Get: $1 = \alpha$

$1 = 1 + \beta n^2 + \gamma n + \delta \Rightarrow \begin{cases} \alpha = 1 \\ \beta = 0 \\ \gamma = 0 \\ \delta = 0 \end{cases}$

and $\boxed{A(n) = 1}$ all n

② Let $\boxed{S_n = n}$ for all $n \in \mathbb{N}$

Get $\alpha = S_0 = 0$

$n = n - 1 + \beta n^2 + \gamma n + \delta$

$0 = \beta n^2 + \gamma n + \delta - 1$

$\Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \\ \delta = 1 \end{cases}$

and $\boxed{D(n) = n}$ for all n

③ Let $\boxed{S_n = n^2}$ for all n

$0 = S_0 = \alpha$

$n^2 = (n-1)^2 + \beta n^2 + \gamma n + \delta$

$n^2 = n^2 - 2n - 1 + \beta n^2 + \gamma n + \delta$

$0 = \beta n^2 + n(-2+\gamma) + (\delta - 1)$

$\Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 2 \\ \delta = -1 \end{cases}$

③ c. d

$$\boxed{n^2 = 2\,C(n) - D(n)}$$

$n^2 = 2C(n) - n$

$$\boxed{C(n) = \frac{n^2 - n}{2}}$$

④ Let $\boxed{S_n = n^3}$ for all n

$0 = S_0 = \alpha$

$n^3 = (n-1)^3 + \beta n^2 + \gamma n + \delta$

$= n^3 - 3n^2 + 3n - 1 + \beta n^2 + \gamma n + \delta$

$0 = n^2(-3+\beta) + n(3+\gamma) + \delta - 1$

$\alpha = 0$
$\beta = 3$
$\gamma = -3$
$\delta = 1$

$$\boxed{n^3 = 3B(n) - 3\,C(n) + D(n)}$$

① + ② + ③ + ④ $\quad A(n) = 1, \; D(n) = 1$

$C(n) = \frac{1}{2}(n^2 + n)$

$B(n) = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{n}{2}\right)$

for S_n

$\alpha = 0$
$\beta = 1$
$\gamma = 0 \quad \delta = 0$

$$\boxed{S_n = \sum_{k=0}^{n} k^2 = B(n) = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{n}{2}\right)}$$

$$S_n = \frac{1}{6}n(n+1)(2n+1)$$

QUESTION 5

DEFINE

$$\Delta^* f(x) = f(x) - f(x-1)$$

FIND formula for

$$\Delta^* x^{\underline{m}}$$

$$x^{\underline{m}} = x(x-1)\cdots\cdots(x-m+1)$$

$$\Delta^*\left(x^{\underline{m}}\right) = x^{\underline{m}} - (x-1)^{\underline{m}}$$

$$= x(x-1)\ldots(x-m+1) - (x-1)\ldots(x-m+1)(x-m$$

$$= (x-1)\cdots\cdots(x-m+1)\big(x - (x-m)\big)$$

$$= (x-1)\ldots(x-m+1)\, m$$

$$= m\,(x-1)^{\underline{m-1}}$$

$$\boxed{\Delta^* x^{\underline{m}} = m\,(x-1)^{\underline{m-1}}}$$

Summation by PARTS

Q6 USE:

- $\sum x^{\underline{m}} \delta x = \dfrac{x^{\underline{m+1}}}{m+1} \quad m \neq -1$

- $\boxed{\sum x^{\underline{-1}} \delta x = H_x}$ • $\boxed{x^{\underline{-1}} = \dfrac{1}{x+1}}$

- $\boxed{\Delta H_x = x^{\underline{-1}} = \dfrac{1}{x+1}}$

- $\Delta x^{\underline{m}} = m x^{\underline{m-1}}$

$$\sum u \Delta v = uv - \sum Ev \Delta u$$

EVALUATE

$$Ev(x) = v(x+1)$$

$$\boxed{\sum \underbrace{x}_{\Delta v} \; \underbrace{H_x}_{u} \; \delta x} = uv - \sum Ev \Delta u$$

CHOICE: $\boxed{\begin{aligned} u(x) &= H_x \\ \Delta v(x) &= x \end{aligned}}$

Evaluate

$\boxed{v(x) = \dfrac{x^{\underline{2}}}{2}}$ $\boxed{Ev = \dfrac{(x+1)^{\underline{2}}}{2}}$ $\boxed{\Delta u = \dfrac{1}{(x+1)}}$

PUT TOGETHER:

$$\sum \underbrace{x}_{\Delta v} \underbrace{H_x}_{u} \delta x = \frac{x^{\underline{2}}}{2} \cdot H_x - \sum \frac{(x+1)^{\underline{2}}}{2} \cdot \frac{1}{(x+1)} \delta x$$

$$= \frac{x^{\underline{2}}}{2} H_x - \sum \frac{(x+1) \cdot x}{2} \cdot \frac{1}{x+1} \delta x$$

$$= \frac{x^{\underline{2}}}{2} H_x - \frac{1}{2} \sum x \delta x$$

$$= \frac{x^{\underline{2}}}{2} H_x - \frac{1}{2} \cdot \frac{x^{\underline{2}}}{2} + C$$

$$= \frac{x^{\underline{2}}}{2} H_x - \frac{1}{4} \frac{x^{\underline{2}}}{2} + C$$