cse547
DISCRETE MATHEMATICS

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LECTURE 4a

CHAPTER 1, Problem 20
SOLUTION

Problem

Use the repertoire method to solve the general **five-parameter recurrence** RF

Solve means FIND the closed formula CF equivalent to following RF

$$
\begin{aligned}
h(1) &= \alpha; \\
h(2n+0) &= 4h(n) + \gamma_0 n + \beta_0; \\
h(2n+1) &= 4h(n) + \gamma_1 n + \beta_1, \text{ for all } n \geq 1.
\end{aligned}
$$

General Form of CF

Our RF for h is a FIVE parameters function and it is a **generalization** of the General Josephus GJ function f considered before

So we guess that now the **general form** of the CF is also a generalization of the one we already proved for GJ , i.e.

General form of CF is

$$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$$

The **Problem 20** asks us to use the repertoire method to prove that CF is **equivalent** to RF

Thinking Time

Solution requires a system of **10 equations** on
$\alpha, \gamma_0, \beta_0, \gamma_1, \beta_1$, A(n), B(n), C(n), D(n), E(n) and accordingly a **5 repertoire functions**

Let's THINK a bit before we embark on quite complicated calculations and without certainty that they would succeed (look at the solution to the **Problem 16** in Lecture 4)

First : we observe that when when $\gamma_0 = \gamma_1 = 0$, we get that teh function h becomes for Generalize Josephus function f below for k = 4:

$f(1) = \alpha, \quad f(2n + j) = kf(n) + \beta_j,$

where $k \geq 2, \ j = 0, 1$ and $n \geq 0$

It seems **worth to examine first** the case $\gamma_0 = \gamma_1 = 0$

GJ f Closed Formula Solution

We **proved** that GJ function f has a relaxed k-representation closed formula

$$\mathbf{f((1, b_{m-1}, ...b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{k}}$$

where β_{b_j} are defined by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \quad ; \qquad j = 0, ..., m-1,$$

for the **relaxed k- radix representation** defined as

$$(\alpha, \beta_{\mathbf{b_{m-1}}}, ..., \beta_{\mathbf{b_0}})_{\mathbf{k}} = \alpha \mathbf{k^m} + \mathbf{k^{m-1}} \beta_{\mathbf{m-1}} + ... + \beta_{\mathbf{b_0}}$$

Special Case of h

Consider now a special case of our h, when $\gamma_0 = \gamma_1 = 0$

We know that it now has a relaxed 4 - representation closed formula

$$h((1, b_{m-1}, ...b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$$

It means that we get

Fact 0 For any $n = (1, b_{m-1}, ...b_1, b_0)_2$,

$$h(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$$

Observe that our general form of CF in this case becomes

$$h(n) = \alpha A(n) + \beta_0 D(n) + \beta_1 E(n)$$

We must have h(n) = h(n), for all n, so from this and **Fact 0** we get the following equation 1 (stated as Fact 1)

Equation 1

Fact 1 For any $n = (1, b_{m-1}, ...b_1, b_0)_2$,

$$\alpha A(n) + \beta_0 D(n) + \beta_1 E(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$$

This provides us with the **Equation 1** for finding our general form of CF

Next Observation

Observe that A(n) in the Original Josephus was proved to be given by a formula

$A(n) = 2^k$, for all $n = 2^k + \ell, \ 0 \leq \ell < 2^k$

So we wonder if we could have a **similar solution** for our A(n)

Special Case of h

We evaluate now few initial values for h in case $\gamma_0 = \gamma_1 = 0$

$$
\begin{aligned}
h(1) &= \alpha; \\
h(2) &= h(2(1)+0) = 4h(1) + \beta_0 \\
&= 4\alpha + \beta_0;
\end{aligned}
$$

$$
\begin{aligned}
h(3) &= h(2(1)+1) = 4h(1) + \beta_1 \\
&= 4\alpha + \beta_1;
\end{aligned}
$$

$$
\begin{aligned}
h(4) &= h(2(2)+0) = 4h(2) + \beta_0 \\
&= 16\alpha + 5\beta_0;
\end{aligned}
$$

Equation 2

It is pretty obvious that we do have a similar formula for A(n) as on the Original Josephus OJ

We write it as the next

Fact 2

For all $n = 2^k + \ell, \;\; 0 \leq \ell < 2^k, \; n \in N - \{0\}$

$$A(n) = 4^k$$

This provides us with the **Equation 2** for finding our general form of CF

Repertoire Method

The proof of **Fact 2** is almost identical to the one in the case of OJ, and for the Problem in Lecture 4, so leave it as an exercise

We have already developed **2 Equations** (as stated in **Facts 1, 2**) so we need now to consider only **3 repertoire functions** to obtain **all Equations** need to solve the problem

Repertoire Function 1

We return now to out **original functions**:

RF: $h(1) = \alpha, h(2n) = 4h(n) + \gamma_0 n + \beta_0,$
$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1,$

CF: $h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

Consider a **first repertoire function** : **h(n) = 1**, for all $n \in N - \{0\}$

We put h(n) = **h(n) = 1**, for all $n \in N - \{0\}$

We have **h(1) = 1**, and h(1)= α, so we get $\alpha = 1$

We now use h(n) = **h(n) = 1**, for all $n \in N - \{0\}$ and evaluate

$h(2n) = 4h(n) + \gamma_0 n + \beta_0$	$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1;$
$1 = 4 + \gamma_0 n + \beta_0$	$1 = 4 + \gamma_1 n + \beta_1$
$0 = (3 + \beta_0) + \gamma_0 n$	$0 = (3 + \beta_1) + \gamma_1 n$

We get $\gamma_0 = \gamma_1 = 0, \quad \beta_0 = \beta_1 = -3$

Solution 1: $\alpha = 1, \quad \gamma_0 = \gamma_1 = 0, \quad \beta_0 = \beta_1 = -3$

Equation 3

The general form of CF is:

$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We put h(n) = **h(n) = 1**, for all $n \in N - \{0\}$, i.e.

$\alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$ = **h(n) = 1**, for all $n \in N - \{0\}$, where $\alpha, \gamma_1, \beta_0, \gamma_2, \beta_1$ already are evaluated in the **Solution 1** as $\alpha = 1, \gamma_0 = \gamma_1 = 0, \beta_0 = \beta_1 = -3$

We get

CF = RF if and only if the following holds

Fact 3 For all $n \in N - \{0\}$,

$$A(n) - 3D(n) - 3E(n) = 1$$

This is our **Equation 3**

Repertoire Function 2

Consider a **repertoire function 2**: **h(n) = n**, for all $n \in N - \{0\}$

We put h(n) = **h(n) = n**, for all $n \in N - \{0\}$

h(1) = α, **h(1)** = 1 and h(n)=**h(n)**, hence $\alpha = 1$

We now use h(n) = **h(n) = n**, for all $n \in N - \{0\}$ and evaluate

$h(2n) = 4h(n) + \gamma_0 n + \beta_0$	$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1$;
$2n = 4n + \gamma_0 n + \beta_0$	$2n + 1 = 4n + \gamma_1 n + \beta_1$
$0 = (\gamma_0 + 2)n + \beta_0$	$0 = (\gamma_1 + 2)n + (\beta_1 - 1)$

We get $\gamma_0 = \gamma_1 = -2,\ \beta_0 = 0,\ \beta_1 = 1$ and

Solution 2: $\alpha = 1,\quad \gamma_0 = \gamma_1 = -2,\ \beta_0 = 0,\ \beta_1 = 1$

Equation 4

CF: $h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We evaluate CF for h(n) = **h(n) = n**, for all $n \in N - \{0\}$ and for the **Solution 2:** $\alpha = 1, \gamma_0 = \gamma_1 = -2,\ \beta_0 = 0,\ \beta_1 = 1$ and get

CF = RF if and only if the following holds

Fact 4 For all $n \in N - \{0\}$

$$A(n) - 2B(n) - 2C(n) + E(n) = n$$

This is our **Equation 4**

Repertoire Function 3

Consider a **repertoire function 3**: $\mathbf{h(n) = n^2}$, for all $n \in N$

We put h(n) = $\mathbf{h(n) = n^2}$, for all $n \in N - \{0\}$

h(1) = α, **h(1)** = 1, hence $\alpha = 1$

$h(2n+0) = 4h(n) + \gamma_0 n + \beta_0$	$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1$;
$(2n)^2 = 4n^2 + \gamma_0 n + \beta_0$	$(2n+1)^2 = 4n^2 + \gamma_1 n + \beta_1$
$\not{4}n^2 = \not{4}n^2 + \gamma_0 n + \beta_0$	$4n^2 + 4n + 1 = 4n^2 + \gamma_1 n + \beta_1$
$0 = \gamma_0 n + \beta_0$	$0 = (\gamma_1 - 4)n + (\beta_1 - 1)$

We get $\gamma_0 = 0,\ \gamma_1 = 4,\ \beta_0 = 0,\ \beta_1 = 1$ and

Solution 3: $\alpha = 1,\ \gamma_0 = 0,\ \gamma_1 = 4,\ \beta_0 = 0,\ \beta_1 = 1$

Equation 5

CF: $h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We evaluate CF for h(n) = $\mathbf{h}(\mathbf{n}) = \mathbf{n}^2$, for all $n \in N - \{0\}$

and for the **Solution 3:**

$\alpha = 1,\ \gamma_0 = 0,\ \gamma_1 = 4,\ \beta_0 = 0,\ \beta_1 = 1$

We get CF = RF if and only if the following holds

Fact 5 For all $n \in N - \{0\}$

$$A(n) + 4C(n) + E(n) = n^2$$

This is our **Equation 5**

Repertoire Method: System of Equations

We obtained the following system of **5 equations** on A(n), B(n), C(n), D(n), E(n)

1. $\alpha A(n) + \beta_0 D(n) + \beta_1 E(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$

2. $A(n) = 4^k$

3. A(n) - 3D(n) - 3E(n) = 1

4. A(n) - 2B(n) - 2 C(n) + E(n) = n

5. $A(n) + 4C(n) + E(n) = n^2$

We solve it and put the solution into

$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

System of Equations Solution

First solution formulas

$$A(n) = 4^k$$

$$B(n) = \frac{A(n) - 3C(n) - 1}{3} \quad \text{(from } \textbf{3.}\text{)}$$

$$D(n) = \frac{3A(n) + 3C(n) - n^2 - 2n}{4}$$

$$E(n) = \frac{n^2 - A(n) - C(n)}{4} \quad \text{(from } \textbf{5.}\text{)}$$

$$C(n) = ((\alpha\beta_{b_{m-1}} ... \beta_{b_1}\beta_{b_0})_4 - \alpha \cdot 4^k - \beta_0(1 - 4^k)/3)/(\beta_0 + \beta_1)$$

Problem 20 Solution

After substitution we obtain:

$$A(n) = 4^k$$

$$B(n) = \frac{4^k - 3C(n) - 1}{3}$$

$$D(n) = \frac{3\ 4^k + 3C(n) - n^2 - 2n}{4}$$

$$E(n) = \frac{n^2 - 4^k - C(n)}{4}$$

$$C(n) = ((\alpha\beta_{b_{m-1}}...\beta_{b_1}\beta_{b_0})_4 - \alpha \cdot 4^k - \beta_0(1 - 4^k)/3)/(\beta_0 + \beta_1)$$

General Remark

Observe that the use of the **Relaxed k -Radix Representation** partial solution, i.e. the **equation 1** as well as proving the formula for A(n), i.e. the **equation 2** were essential for the **solution of the problem**

Like in Ch1 **Problem 16** - the other typical repertoire functions like $h(n) = n^3$ etc . **FAIL**, i.e. lead to contradictions - easy and left for you to evaluate!

Read carefully published Solutions to Ch1**Problem 16**, which is, in fact an easier case of **Problem 20** just presented in a FULL DETAIL

Observe also that we have **proved** all **Relaxed k -Radix Representation** formulas needed and you have to KNOW these proofs for your tests