cse547, math547
DISCRETE MATHEMATICS

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LECTURE 8a

CHAPTER 2
SUMS

Part 1: Introduction - Lecture 5

Part 2: Sums and Recurrences (1) - Lecture 5

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CHAPTER 2
SUMS

Part 3: Multiple Sums (3) General Methods - Lecture 8a

SUMS
GENERAL METHODS

Method 0

PROBLEM : Find a Closed Formula for

$$\square_n = \sum_{k=0}^{n} k^2$$

Method 0 : Look it up p.72 of
CRS Standard Mathematical Tables and give answer:

$$\square_n = \frac{n(n+1)(2n+1)}{6}$$

Other references:
Handbook of Math functions: Abramowitz, Stequr
Handbook of Integer Sequences, Sloane

Method 1

Guess the answer and prove by **Mathematical Induction**

$$\square_n = \frac{n(n+\frac{1}{2})(n+1)}{3} = \sum_{k=0}^{n} k^2$$

Re-write as

$$\square_0 = 0, \quad \square_n = \square_{n-1} + n^2$$

Use **inductive assumption** for $n := n-1$

$3\square_n = (n-1)(n-1+\frac{1}{2})n + 3n^2 = (n-1)(n-\frac{1}{2})n + 3n^2$

$= n^3 - \frac{3}{2}n^2 + \frac{1}{2}n + 3n^2 = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n$

$= n(n^2 + \frac{3}{2}n + \frac{1}{2}) = n(n+\frac{1}{2})(n+1)$

Method 2: Perturbation Method

Perturbation Method is a method that often allows us to evaluate a CF form for a certain sums

The **idea** is to start with an **unknown sum** and call it S_n:

$$S_n = \sum_{k=0}^{n} a_k$$

Then we re-write S_{n+1} in **two ways**, by splitting off both its last term a_{n+1} and its first term a_0:

$$\begin{aligned} S_n + a_{n+1} &= a_0 + \sum_{k=1}^{n+1} a_k \quad \text{put } k{:=}\ k+1 \\ &= a_0 + \sum_{1 \leq k+1 \leq n+1} a_{k+1} = a_0 + \sum_{0 \leq k \leq n} a_{k+1} \\ &= a_0 + \sum_{k=0}^{n} a_{k+1} \end{aligned}$$

Method 2: Perturbation Method

We get a formula:

$$S_n + a_{n+1} = a_0 + \sum_{k=0}^{n} a_{k+1}$$

Goal of the Perturbation Method is to **work** on the **last sum**

$\sum_{k=0}^{n} a_{k+1}$ and try to express it on terms of S_n

If it **works** and if we get a **multiple** of S_n we **solve** the equation on S_n and obtain the closed formula CF for the **original sum**

If it **does not work** - we look for another method

Method 2: Perturb the Sum

Method 2: Perturb the Sum $\square_n = \sum_{k=0}^{n} k^2$

We use the perturbation formula: for $a_0 = 0$ in this case

$$S_n + a_{n+1} = a_0 + \sum_{k=0}^{n} a_{k+1}$$

$$\square_n + (n+1)^2 = \sum_{k=0}^{n} (k+1)^2 = \sum_{k=0}^{n} (k^2 + 2k + 1)$$

$$= \sum_{k=0}^{n} k^2 + 2\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1$$

$$= \square_n + 2\sum_{k=0}^{n} k + (n+1)$$

Nice calculation but NO RESULT for $\square_n$!

Method failed

Method 2: Perturb the Sum

Nevertheless we get something:

$$(n+1)^2 = 2\sum_{k=0}^{n} k + (n+1)$$

$$2\sum_{k=0}^{n} k = (n+1)^2 - (n+1)$$

$$2\sum_{k=0}^{n} k = (n+1)(n+1-1)$$

BONUS : $$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Method 2: Perturb the Sum

Back to our problem: $\boxed{\square_n = \sum_{k=0}^{n} k^2}$

IDEA: use perturbation for

$\boxed{⧉_n = \sum_{k=0}^{n} k^3}$ to get $\boxed{\square_n}$ as we did for $\boxed{\sum_{k=0}^{n} k}$

We use as before the perturbation formula: for $a_0 = 0$ in this case

$$S_n + a_{n+1} \;=\; a_0 \;+ \sum_{k=0}^{n} a_{k+1}$$

We evaluate

$$⧉_n + (n+1)^3 = \sum_{k=0}^{n} (k+1)^3 = \sum_{k=0}^{n} (k^3 + 3k^2 + 3k + 1)$$

Reminder: $\boxed{(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3}$

Method 2: Perturb the Sum

Method 2: Perturb the Sum

We evaluate now

$$\text{⧉}_n + (n+1)^3 = \textstyle\sum_{k=0}^{n}(k^3+3k^2+3k+1)$$

$$= \sum_{k=0}^{n} k^3 + 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1$$

$$= \text{⧉}_n + \boxed{3\Box_n} + 3\frac{n(n+1)}{2} + (n+1) \qquad \text{Got it!}$$

We have now

$$(n+1)^3 = \boxed{3\Box_n} + 3\frac{n(n+1)}{2} + (n+1)$$

$$\boxed{3\Box_n} = (n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)$$

$$= (n+1)((n+1)^2 - \tfrac{3}{2}n - 1) = (n+1)(n^2 + \tfrac{1}{2}n)$$

$$\Box_n = \frac{(n+1)(n+\frac{1}{2})n}{3}$$

Method 3: Repertoire

Method 3: Build a **repertoire**

Back to our problem: **evaluate** $\square_n = \sum_{k=0}^{n} k^2$

To solve it we now generalize the recursive formula

(1) RF: $R_0 = \alpha, \; R_n = R_{n-1} + \beta + \gamma n$

we have used to evaluate the sum $\sum_{k=0}^{n}(a+bk)$ to a formula

(2) RF: $R_0 = \alpha, \; R_n = R_{n-1} + \beta + \gamma n + \boxed{\delta n^2}$

which we **now use to evaluate** the sum $\sum_{k=0}^{n}(a+nk^2)$

Method 3: Repertoire

Now the general form of the closed form CF formula is

$$CF: \quad R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$$

Observe that when $\delta = 0$, we get the case **(1)** and as we have already **evaluated before**

$$A(n) = 1, \quad B(n) = n, \quad C(n) = \frac{n^2+n}{2}$$

General closed formula CF becomes

$$\textbf{(3)} \qquad R_n = \alpha + n\beta + \frac{(n^2+n)}{2}\gamma + D(n)\delta$$

Method 3: Repertoire

We need to evaluate $\boxed{D(n)}$

We use a **repertoire function** $\boxed{R_n = n^3, \text{ for all } n \in N}$

to evaluate $\alpha, \beta, \gamma, \delta$ (if exists)

Our recurrence

(2) $\boxed{\text{RF:} \quad R_0 = \alpha, \; R_n = R_{n-1} + \beta + \gamma n + \boxed{\delta n^2}}$

We evaluate $R_0 = R_0 = 0$ iff $\boxed{\alpha = 0}$

We set $R_n = R_n = n^3$, for all $n \in N$ and evaluate

$n^3 = (n-1)^3 + \beta + \gamma n + \delta n^2$ for all $n \in N$

$0 = n^2(\delta - 3) + n(\gamma + 3) + (\beta - 1)$ for all $n \in N$

This is possible only when

$$\boxed{\delta = 3}, \quad \boxed{\gamma = -3}, \quad \boxed{\beta = 1}$$

Method 3: Repertoire

Our closed CF formula is

$$CF: \quad R_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma + D(n)\delta$$

For $R_n = n^3$ and $\alpha = 0,\ \beta = 1,\ \gamma = -3,\ \delta = 3$

it becomes

$$n^3 = 0 + n - \frac{3}{2}(n^2+n) + \boxed{3D(n)}$$

$$\boxed{3D(n)} = n^3 - n + \frac{3}{2}(n^2+n)$$

$$= n(n^2 + \frac{3}{2}n + \frac{1}{2}) = \boxed{n(n+\frac{1}{2})(n+1)} \quad \text{and}$$

$$\boxed{D(n) = \frac{n(n+\frac{1}{2})(n+1)}{3}}$$

$$CF: \quad R_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma + \frac{n(n+\frac{1}{2})(n+1)}{3}\delta$$

Method 3: Repertoire

Observe that our sum

$$\square_n = \sum_{k=0}^{n} k^2$$

written as

$$\square_0 = 0, \ \square_n = \square_{n-1} + n^2$$

is a special case of

$$R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n + \delta n^2$$

for $\alpha = 0, \ \beta = 0, \ \gamma = 0, \ \delta = 1$

and closed formula

$$CF: \quad R_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma + \frac{n(n+\frac{1}{2})(n+1)}{3}\delta$$

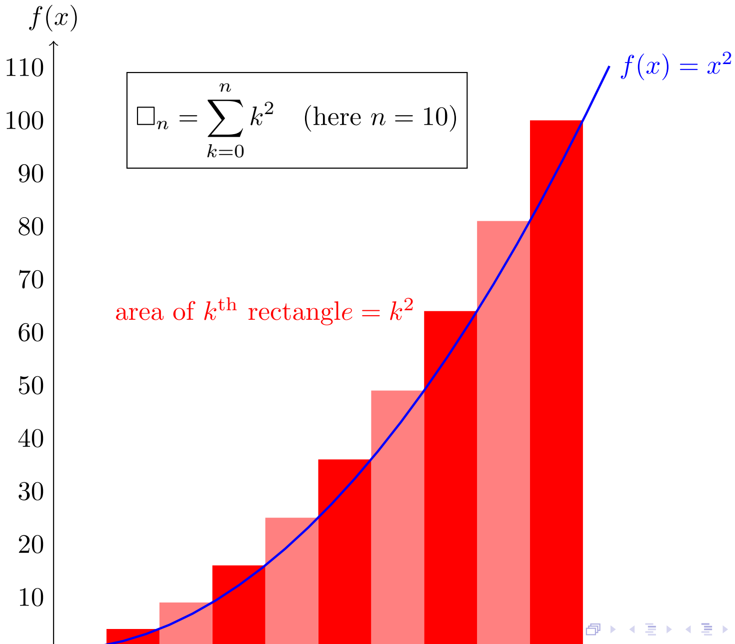
becomes

$$R_n = \square_n = \frac{n(n+\frac{1}{2})(n+1)}{3}$$

and

$$\sum_{k=0}^{n} k^2 = \frac{n(n+\frac{1}{2})(n+1)}{3}$$

Method 4: Replace Sums by Integrals



Method 4: Replace Sums by Integrals

Area under the curve up to n is $\int_0^n x^2\,dx = \left[\frac{x^3}{3}\right]_0^n = \frac{n^3}{3}$

Hence

$$\boxed{\square_n \approx \frac{n^3}{3}.}$$

More precisely, letting E_n denote the error, we have

$$\boxed{\square_n = \frac{n^3}{3} + E_n}$$

Method 4: Replace Sums by Integrals

$$\boxed{\text{Error} = E_n = \Box_n - \frac{1}{3}n^3}$$

We want a recursive formula for E_n

$$\begin{aligned} E_{n-1} &= \Box_{n-1} - \frac{1}{3}(n-1)^3 \\ &= \Box_{n-1} - \frac{1}{3}(n^3 - 3n^2 + 3n - 1) \\ &= \Box_{n-1} - \frac{1}{3}n^3 + n^2 - \frac{3}{3}n + \frac{1}{3} \\ &= \Box_{n-1} + n^2 - \frac{1}{3}n^3 + \boxed{\frac{1}{3} - n} \end{aligned}$$

Method 4: Replace Sums by Integrals

We have that $E_n = \Box_n - \frac{1}{3}n^3$ and obtained

$$E_{n-1} = \Box_{n-1} + n^2 - \frac{1}{3}n^3 + \frac{1}{3} - n$$

Since $\Box_n = \Box_{n-1} + n^2$, we have

$$\begin{aligned} E_n &= \Box_n - \frac{1}{3}n^3 \\ &= \Box_{n-1} + n^2 - \frac{1}{3}n^3 \\ &= \underbrace{\Box_{n-1} + n^2 - \frac{1}{3}n^3 + \left(\frac{1}{3} - n\right)}_{E_{n-1}} - \left(\frac{1}{3} - n\right) \\ &= E_{n-1} + n - \frac{1}{3} \end{aligned}$$

Method 4: Replace Sums by Integrals

Hence

$$\boxed{E_n = E_{n-1} + n - \frac{1}{3}, \quad E_0 = 0} \qquad \text{(RF Formula)}$$

and

$$\boxed{E_n = \sum_{k=1}^{n} \left(k - \frac{1}{3}\right)} \qquad \text{(SUM Formula)}$$

Method 4: Replace Sums by Integrals

We evaluate:

$$\boxed{E_n} = \sum_{k=1}^{n} \left(k - \frac{1}{3}\right)$$

$$= \sum_{k=1}^{n} k - \frac{1}{3} \sum_{k=1}^{n} 1$$

$$= \boxed{\frac{n(n+1)}{2} - \frac{1}{3} n}$$

$$E_n = \frac{n(n+1)}{2} - \frac{1}{3} n$$

Method 4: Replace Sums by Integrals

Evaluate:

$$\begin{aligned}
\boxed{\square_n} &= E_n + \frac{n^3}{3} \\
&= \frac{n(n+1)}{2} - \frac{1}{3}n + \frac{n^3}{3} \\
&= \frac{3n^2 + 3n - 2n + 2n^3}{6} \\
&= \boxed{\frac{2n^3 + 3n^2 + n}{6}.}
\end{aligned}$$

Method 4: Replace Sums by Integrals

Check:

$$\boxed{\square_n = \frac{n\left(n+\frac{1}{2}\right)(n+1)}{3}}$$

$$= \frac{n^3+\frac{1}{2}n^2+n^2+\frac{1}{2}n}{3} = \frac{2n^3+n^2+2n^2+n}{6}$$

$$= \boxed{\frac{2n^3+3n^2+n}{6}.}$$

Method 5: Expand and Contract

Method:

Replace the original single sum by a seemingly more complicated DOUBLE sum (**expend**) that can be in turn simplified **(contract)**

Example of replacement

$$\boxed{\sum_{k=1}^{n} k^2 = \sum_{1 \le j \le k \le n} k}$$

Proof on the next slide

Method 5: Expand and Contract

$$\boxed{\sum_{k=1}^{n} k^2} = \sum_{k=1}^{n} k \cdot k$$

$$= \sum_{k=1}^{n} k \underbrace{\left(\sum_{j=1}^{k} 1\right)}_{k=\sum_{j=1}^{k} 1}$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{k} k$$

$$= \sum_{j=1}^{k} \sum_{k=1}^{n} k$$

$$= \boxed{\sum_{1 \leq j \leq k \leq n} k,}$$

Method 5: Expand and Contract

We have used in the proof a following **property** (yet to be proved!)

$$(1 \leq j \leq k) \cap (1 \leq k \leq n) \equiv 1 \leq j \leq k \leq n$$

Observe that we have two possibilities for j and k :
$j > k$, or $j \leq k$.

Note that when $j > k$, the sum $\sum_{j=1}^{k} 1$ does not exist (DNE) so this case is **impossible**

Method 5: Expand and Contract

The case $j \leq k$ is obvious, i.e. we have that

$$(1 \leq j \leq k) \cap (1 \leq k \leq n) \equiv (1 \leq j \leq k \leq n) \quad \text{when} \quad j \leq k$$

We have hence **proved** the following general property of **changing the limits** of summation for future use

$$\boxed{\sum_{j=1}^{k} \sum_{k=1}^{n} a_{ij} = \sum_{1 \leq j \leq k \leq n} a_{ij}} \qquad \text{(PROPERTY)}$$

Method 5: Expand and Contract

$$\square_n = \sum_{k=1}^{n} k^2 = \sum_{1 \le j \le k \le n} k$$

$$= \underbrace{\sum_{j=1}^{n} \left(\sum_{k=j}^{n} k \right)}_{\substack{1 \le j \le k \le n \\ \equiv (1 \le j \le n) \\ \cap (j \le k \le n)}}$$

$$= \sum_{j=1}^{n} \frac{(j+n)(n-j+1)}{2}$$

Method 5: Expand and Contract

$$\square_n = \sum_{k=1}^{n} k^2 = \sum_{1 \le j \le k \le n} k$$

$$= \frac{1}{2} \sum_{j=1}^{n} \left(n(n+1) + j - j^2 \right)$$

$$= \frac{1}{2} n^2(n+1) + \frac{1}{2} \sum_{j=1}^{n} j - \frac{1}{2} \sum_{j=1}^{n} j^2$$

$$= \frac{1}{2} n^2(n+1) + \frac{1}{4} n(n+1) - \frac{1}{2} \boxed{\sum_{k=1}^{n} k^2}$$

$$= \frac{1}{2} n \left(n + \frac{1}{2} \right) (n+1) - \frac{1}{2} \square_n.$$

Method 5: Expand and Contract

Hence

$$\frac{3}{2}\Box_n = \frac{1}{2}n\left(n+\frac{1}{2}\right)(n+1),$$

i.e.

$$\boxed{\Box_n = \frac{n\left(n+\frac{1}{2}\right)(n+1)}{3}}$$

Note that in the above we used

$$\sum_{j=1}^{n} n(n+1) = n(n+1)\sum_{j=1}^{n} 1 = n^2(n+1).$$

Method 5: Expand and Contract

Prove by Method 5 the following property

$$\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 = 2 \cdot \sum_{1 \le j \le k \le n} j \cdot k$$

We use this property in Chapter 2, problem 15

First we evaluate $\sum_{k=1}^{n} k^3 + (n+1)^3$ in terms of $\sum_{k=1}^{n} k^2$ and $\sum_{1 \le j \le k \le n} j \cdot k$

$$\sum_{k=1}^{n} k^3 + (n+1)^3 = \sum_{k=0}^{n-1} (k+1)^3 + (n+1)^3$$

$$= \sum_{k=0}^{n} (k+1)^3$$

$$= 1 + \sum_{k=1}^{n} (k+1)^3$$

Method 5: Expand and Contract

$$\sum_{k=1}^{n} k^3 + (n+1)^3 = 1 + \underbrace{\sum_{k=1}^{n} k^2(k+1) + \sum_{k=1}^{n} (2k+1)(k+1)}_{(k+1)^3=(k^2+2k+1)(k+1)=k^2(k+1)+(2k+1)(k+1)}$$

$$= 1 + \sum_{k=1}^{n} k \cdot \underbrace{k(k+1)}_{=2\sum_{j=1}^{k} j} + \sum_{k=1}^{n} (2k+1)(k+1)$$

$$= 1 + \sum_{k=1}^{n} k \left(2 \sum_{j=1}^{k} j \right) + \sum_{k=1}^{n} (2k^2+3k+1)$$

$$= 1 + 2 \sum_{k=1}^{n} \sum_{j=1}^{k} k \cdot j + 2 \sum_{k=1}^{n} k^2 + 3 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 1 + 2 \sum_{1 \le j \le k \le n} k \cdot j + \frac{3}{2} n(n+1) + n + 2 \sum_{k=1}^{n} k^2$$

Method 5: Expand and Contract

Adding the factor $\sum_{k=1}^{n} k^2 - (n+1)^3$ to both sides, we get

$$\boxed{\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2} = 2 \sum_{1 \leq j \leq k \leq n} k \cdot j + \underbrace{3 \sum_{k=1}^{n} k^2}_{= n(n+1/2)(n+1)}$$

$$+ 3 \cdot \frac{n(n+1)}{2} + (n+1) - (n+1)^3$$

$$= \boxed{2 \sum_{1 \leq j \leq k \leq n} k \cdot j}$$

$$+ \boxed{n\left(n + \frac{1}{2}\right)(n+1) + \frac{3}{2} n(n+1) + (n+1) - (n+1)^3}$$

Method 5 : Expand and Contract

We evaluate now the last factor

$$
\begin{aligned}
n\left(n+\frac{1}{2}\right)(n+1)+\frac{3}{2}n(n+1)+(n+1)-(n+1)^3 &= (n+1)\Big[n\left(n+\frac{1}{2}\right)+\frac{3}{2}n \\
&\quad +1-(n+1)^2\Big] \\
&= (n+1)\Big(n^2+\frac{1}{2}n+\frac{3}{2}n \\
&\quad +1-n^2-2n-1\Big) \\
&= (n+1)(2n-2n) \\
&= 0
\end{aligned}
$$

Method 5: Expand and Contract

Hence we have **proved** that

$$\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 = 2 \sum_{1 \le j \le k \le n} k \cdot j$$

Note that we used in the proof already **proved** property

$$\sum_{k=1}^{n} \sum_{j=1}^{k} k \cdot j = \sum_{j=1}^{k} \sum_{k=1}^{n} k \cdot j = \sum_{1 \le j \le k \le n} k \cdot j$$

Short Proof

$$\boxed{\sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k^3 = 2 \sum_{1 \le j \le k \le n} k \cdot j}$$

Here is a **short proof:**

$$\boxed{\sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k^3} = 0^2 + 0^3 + \sum_{k=1}^{n} (k^2 + k^3)$$

$$= \sum_{k=1}^{n} \underbrace{k^2(k+1)}_{\substack{\text{use} \\ \frac{k(k+1)}{2} = \sum_{j=1}^{k} j}}$$

Short Proof

$$\boxed{\sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k^3} = \sum_{k=1}^{n} 2k \cdot \frac{k(k+1)}{2}$$

$$= 2 \sum_{k=1}^{n} \underbrace{k}_{\substack{\text{const.} \\ \text{on } j}} \cdot \sum_{j=1}^{k} j$$

$$= 2 \underbrace{\sum_{k=1}^{n} \sum_{j=1}^{k} k \cdot j}_{\substack{\text{use } \sum_{k=1}^{n} \sum_{j=1}^{k} a_{jk} \\ = \sum_{1 \le j \le k \le n} a_{jk}}}$$

$$= \boxed{2 \sum_{1 \le j \le k \le n} k \cdot j}$$