cse547, math547
DISCRETE MATHEMATICS

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LECTURE 9b

CHAPTER 2
SUMS

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Part 4: Finite and Infinite Calculus (2) - Lecture 9b

Definite Integral and Definite Sum

Infinite Calculus: DEFINITE INTEGRAL

$$\int_a^b g(x)dx = f(x)\Big|_a^b = f(b) - f(a)$$ where $$f'(x) = g(x)$$

Finite Calculus: DEFINITE SUM

$$\sum_a^b g(x)\delta_x = f(x)\Big|_a^b = f(b) - f(a)$$

where $\Delta f(x) = g(x)$

Definite Sum Properties

Take $b < a$ and evaluate

$$(1) \quad \sum_a^b g(x)\delta_x = f(b) - f(a)$$
$$= -(f(b) - f(a))$$
$$= -\sum_b^a g(x)\delta_x$$

$$(2) \quad \sum_a^b g(x)\delta_x + \sum_b^c g(x)\delta_x = \sum_a^c g(x)\delta_x$$

For all $a, b, c \in Z$

Exercise

Exercise:

FIND the sum: $\sum_{k=0}^{n-1} k^{\underline{m}}$

Hint: use

$$\underbrace{\sum_{k=a}^{b-1} g(k)}_{\text{SUM}} \stackrel{\text{Thm}}{=} \sum_a^b g(x)\delta_x$$

where $g(x) = \Delta f(x) = f(x+1) - f(x)$

$$\Delta x^{\underline{m}} = m x^{\underline{m-1}}$$

$$\sum x^{\underline{m}} \delta_x = \frac{x^{\underline{m+1}}}{m+1}$$

Exercise

SOLUTION:

$$\sum_{k=0}^{n-1} k^{\underline{m}} \overset{(thm)}{=} \sum_{k=0}^{n} x^{\underline{m}} \delta_x = \frac{x^{\underline{m+1}}}{m+1}\Big|_0^n = \frac{n^{\underline{m+1}}}{m+1}$$

used: $\Delta\left(\frac{x^{\underline{m+1}}}{m+1}\right) = \frac{1}{m+1}\Delta x^{\underline{m+1}} = \frac{m+1}{m+1} \cdot x^{\underline{m}} = x^{\underline{m}}$

We proved

$$\sum_{0 \le k < n} k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\Big|_0^n = \frac{n^{\underline{m+1}}}{m+1} \qquad n, m \ge 0$$

Exercise

Exercise: Use ”integration” to evaluate

$$\sum_{k=0}^{n-1} k = \sum_{0 \le k < n} k$$

Observe that

$$\boxed{k^{\underline{1}} = k} \quad \text{as} \quad \boxed{x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)}$$

$$\begin{aligned} \textstyle\sum_{k=0}^{n-1} k &= \textstyle\sum_{k=0}^{n-1} k^{\underline{1}} = \sum_{0 \le k < n} k^{\underline{1}} \\ &\overset{\text{Thm}}{=} \textstyle\sum_{0}^{n} x^{\underline{1}} \delta_x = \frac{x^{\underline{2}}}{2}\Big|_0^n \\ &= \frac{n^{\underline{2}}}{2} = \frac{n(n-1)}{2} \end{aligned}$$

because $n^{\underline{2}} = n \cdots (n-2+1) = n(n-1)$ and hence

$$\boxed{\sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}}$$

Useful Fact

FACT 1 $$k^2 = k^{\underline{2}} + k^{\underline{1}}$$

Proof: $x^{\underline{m}} = x(x-1)\cdots(x-m+1)$

$$\begin{aligned} k^{\underline{2}} &= k(k-2+1) = k(k-1) \\ k^{\underline{1}} &= k \\ k^{\underline{2}} + k^{\underline{1}} &= k(k-1) + k = k(k-1+1) = k^2 \end{aligned}$$

Exercise

Evaluate

$$\boxed{\sum_{k=0}^{n-1} k^2}$$

Hint: use $\boxed{\text{Thm}}$ and $\boxed{\text{FACT1}}$

$$
\begin{aligned}
\sum_{k=0}^{n-1} k^2 &\overset{F1}{=} \sum_{0\le k<n}(k^{\underline{2}}+k^{\underline{1}}) = \sum_{0\le k<n} k^{\underline{2}} + \sum_{0\le k<n} k^{\underline{1}} \\
&\overset{\text{Thm}}{=} \sum_0^n x^{\underline{2}}\delta_x + \sum_0^n x^{\underline{1}}\delta_x \\
&= \frac{x^{\underline{3}}}{3}\Big|_0^n + \frac{x^{\underline{2}}}{2}\Big|_0^n = \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} \\
&= \frac{1}{3}(n(n-1)(n-2)) + \frac{1}{2}(n(n-1)) = \frac{1}{3}n(n-\frac{1}{2})(n-1) \\
&\quad (\frac{1}{3}n(n-1)(n-2+\frac{3}{2})
\end{aligned}
$$

Exercise

FACT2 $$\boxed{k^3 = k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}}$$

Prove it as an exercise and use it to **evaluate** a not trivial sum:

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3$$

$$\begin{aligned}
\textstyle\sum_{k=1}^{n} k^3 &\overset{F1}{=} \textstyle\sum_{k=1}^{n} (k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}) \\
&= \textstyle\sum_{k=1}^{n+1} k^{\underline{3}}\delta_k + 3\sum_{k=1}^{n+1} k^{\underline{2}}\delta_k + \sum_{k=1}^{n+1} k^{\underline{1}}\delta_k \\
&= \tfrac{k^{\underline{4}}}{4} + 3\tfrac{k^{\underline{3}}}{3} + \tfrac{k^{\underline{2}}}{2}\Big|_1^{n+1}
\end{aligned}$$

$$\boxed{\sum_{a \le k < b} k^3 = \frac{k^{\underline{4}}}{4} + 3\frac{k^{\underline{3}}}{3} + \frac{k^{\underline{2}}}{2}\Bigg|_a^b}$$

Simple Problem

PROVE

$$\boxed{(x+y)^{\underline{2}} = x^{\underline{2}} + 2x^{\underline{1}}y^{\underline{1}} + y^{\underline{2}}}$$

$$\begin{aligned} x^{\underline{3}} &= x(x-1)(x-2) \\ x^{\underline{2}} &= x(x-1) \\ x^{\underline{1}} &= x \\ x^{\underline{0}} &= 1 \end{aligned}$$

$$\begin{aligned} x^{\underline{2}} &= \frac{x^{\underline{3}}}{x-2} \\ x^{\underline{1}} &= \frac{x^{\underline{2}}}{x-1} \\ x^{\underline{0}} &= \frac{x^{\underline{1}}}{x} \end{aligned}$$

Negative Exponent Falling Powers

Definition of **negative** exponent falling powers

$$x^{\underline{-1}} = \frac{1}{x+1}$$

$$x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$$

$$x^{\underline{-3}} = \frac{1}{(x+1)(x+2)(x+3)}$$

General:

$$x^{\underline{-m}} = \frac{1}{(x+1)(x+2)\cdots(x+m)} \qquad m > 0$$

Problems

Prove:

$$\boxed{x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}}$$

Prove: for $m < 0$

$$\boxed{\Delta x^{\underline{m}} = m x^{\underline{m-1}}}$$

Example

Example:

$$\boxed{\Delta x^{\underline{-2}}} = \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)}$$

$$= \frac{(x+1)-(x+3)}{(x+1)(x+2)(x+3)}$$

$$= \boxed{-2x^{\underline{-3}}}$$

Fact:

$$\boxed{\sum_a^b x^{\underline{m}} \delta x = \frac{x^{\underline{m+1}}}{m+1}\Big|_a^b} \quad \text{all } m \neq -1$$

What about case $m = -1$?

Example

Case $m = -1$:

Infinite Integral

$$\int_a^b x^{-1} dx = \int_a^b \frac{1}{x} dx = \ln|x| \Big|_a^b$$

We want to have a **finite analog:**

$$x^{\underline{-1}} = \frac{1}{x+1} \quad \Big| \quad \Delta f = f(x+1) - f(x)$$

Take:

$$\boxed{f(x)} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{x} = \boxed{\sum_{k=1}^{x} \frac{1}{k} = H_x}$$

$$\Delta f(x) = \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{x} + \frac{1}{x+1}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{x}\right)$$

$$= \frac{1}{x+1} = \Delta H_x$$

We proved: $\Delta H_x = \dfrac{1}{x+1}$

Example

Case $m = -1$

$$\sum_a^b x^{\underline{-1}} \delta x = \sum_a^b \frac{1}{x+1} \delta x = H_x \Big|_a^b$$

We prove (Chapter 9) that for large x

$$H_x - \ln x \approx 0.577 + \frac{1}{2x}$$

$H_x \sim \ln x$ as do $\int_a^b$ and $\sum_a^b$

Falling Powers Theorem

Theorem: Sums of **falling** powers

$$\sum_a^b x^{\underline{m}} \delta x = \begin{cases} \left. \dfrac{x^{\underline{m+1}}}{m+1} \right|_a^b & m \neq -1 \\ \left. H_x \right|_a^b & m = -1 \end{cases}$$

all $m \in \mathbb{Z}$

and $\int_a^b \frac{1}{x} dx = \ln|x| \Big|_a^b$ is **similar** to $\sum_a^b x^{\underline{-1}} = H_x \Big|_a^b$

More Similarities

We know $(e^x)' = e^x$, i.e.

$$\boxed{De^x = e^x} \qquad Df = f \quad \text{when} \quad f(x) = e^x$$

Question

Can we have a function f that has a **similar** property for Δ?

i.e. a function f such that $\boxed{\Delta f(x) = f(x)}$

$$\Delta f(x) = f(x+1) - f(x) = f(x)$$

Answer: Any f such that: $\boxed{f(x+1) = 2f(x)}$ Recurrence!

Example

Example of solution:

$$\boxed{f(x) = 2^x}$$

$$\begin{aligned} f(x+1) - f(x) &= 2^{x+1} - 2^x \\ &= 2 \cdot 2^x - 2^x \\ &= 2^x = f(x) \end{aligned}$$

We proved: $\boxed{\Delta(2^x) = 2^x}$ $\quad (e^x)' = e^x$

Find a formula for Δf, where $\boxed{f(x) = c^x}$ for $c \in N^+$

$$\begin{aligned} \Delta(c^x) = c^{x+1} - c^x &= c \cdot c^x - c^x \\ &= c^x(c-1) \end{aligned}$$

We proved: $\Delta(c^x) = c^x(c-1)$

Difference

Difference:

$$\boxed{\Delta(c^x) = (c-1)c^x}$$ $c \in N^+$ "derivative"

$$\boxed{\sum_a^b c^x \delta x = \frac{c^x}{c-1}\Big|_a^b \qquad c \neq 1}$$ "antiderivative"

anti-difference

Geometric Progression

We now prove:
Theorem: Geometric Progression

$$\boxed{\sum_{a\le k<b} c^k} = \sum\nolimits_a^b c^x \delta x$$

$$= \frac{c^x}{c-1}\Big|_a^b = \boxed{\frac{c^b - c^a}{c-1}} \qquad c \neq 1$$

General Formula for Geometric Progression

$$\boxed{\sum_{a\le k<b} c^k = \frac{c^b - c^a}{c-1}} \qquad c \neq 1$$

$$\sum_{k=a}^{b-1} c^k = \frac{c^b - c^a}{c-1}$$

Chain Rule

Infinite: "chain rule"

$$Df(g(x)) = Df \cdot Dg(x)$$

Finite: no such rule

Can't relate $\Delta f(g(x))$ to $\Delta g(x)$

Integration by Parts

Infinite

$$\boxed{D(uv) = uDv + vDu}$$

Integration by parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Can we have an **analog** for Δ?

Integration by Parts

Can we have

$$\Delta(uv) = u\Delta v + v\Delta u$$

and $$\sum u\,\delta v = uv - \sum \boxed{v}\,\delta u?$$

change here

Not exactly, but close!

Evaluate:

$$\begin{aligned}
\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\
&= u(x+1)v(x+1) \underline{- u(x)v(x+1) + u(x)v(x+1)} - u(x)v(x) \\
&= u(x)\underline{v(x+1)} - u(x)\underline{v(x)} + \underline{u(x+1)}v(x+1) - \underline{u(x)}v(x+1) \\
&= u(x)(\underline{v(x+1) - v(x)}) + v(x+1)(\underline{u(x+1) - u(x)}) \\
&= u(x)\Delta v(x) + \boxed{v(x+1)}\,\Delta u(x)
\end{aligned}$$

Summation by Parts

We define a shift operator: $Ev(x) = v(x+1)$

We proved $\Delta(uv) = u\Delta v + Ev\Delta u$

Summation by parts

$$\sum u\,\delta v = uv - \sum Ev\,\delta u$$

$$\sum_a^b u\,\delta v = uv\Big|_a^b - \sum_a^b Ev\,\delta u$$

Summation by Parts

Integration

$$\boxed{\int xe^x\,dx} = xe^x - \int 1\cdot e^x\,dx = \boxed{e^x(x-1)+C}$$

Summation

$$\boxed{\sum x2^x\,\delta x} = x2^x - \sum 1\cdot 2^{x+1}\,\delta x = \boxed{x2^x - 2^{x+1} + C(x)}$$

$$\textit{for}\quad C(x) = C(x+1)$$

Evaluate

$$u(x) = x, \quad v(x) = 2^x, \quad Ev(x) = 2^{x+1}$$

$$\Delta u(x) = 1, \quad \Delta v(x) = 2^x$$

$$\textbf{Fact:}\quad \Delta(2^{x+1}) = 2^{x+1}$$

Summation by Parts

In particular, evaluate:

$$\boxed{\sum_{k=0}^{n} k2^k} = 1\cdot 2^1 + 2\cdot 2^2 + \cdots + n\cdot 2^n$$

$$\sum_{k=0}^{n} k2^k = \sum\nolimits_0^{n+1} x2^x \,\delta x = (x2^x - 2^{x+1})\Big|_0^{n+1}$$

$$= ((n+1)2^{n+1} - 2^{n+2}) - (0\cdot 2^0 - 2)$$

$$= (n+1)2^{n+1} - 2\cdot 2^{n+1} + 2$$

$$= (n+1-2)2^{n+1} + 2 = (n-1)2^{n+1} + 2$$

$$\boxed{\sum_{k=0}^{n} k2^k = (n-1)2^{n+1} + 2}$$

Summation by Parts

Use finite calculus to evaluate:

$$\boxed{\sum_{k=0}^{n-1} kH_k}$$

"sum" by parts

Analog:

$$\begin{aligned}\int x \ln x \, dx &= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}dx \\ &= \frac{x^2}{2}\ln x - \int \frac{x}{2}dx \\ &= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} \\ &= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)\end{aligned}$$

Summation by Parts

Use $$\boxed{\sum u\,\delta v = uv - \sum Ev\,\delta u}$$

$$Ev(x) = v(x+1)$$

$$\sum_{0 \le x, n} xH_x\,\delta x = uv - \sum Ev\,\delta u$$

$$\Delta v(x) = x = x^{\underline{1}}$$

$$v(x) = \frac{x^{\underline{2}}}{2}, \quad v(x+1) = \frac{(x+1)^{\underline{2}}}{2}$$

$$\Delta u(x) = \Delta H_x = x^{\underline{-1}}$$

$$\boxed{v(x) = \frac{x^{\underline{2}}}{2}}$$

$$\boxed{Ev(x) = \frac{(x+1)^{\underline{2}}}{2}}$$

$$\boxed{u(x) = H_x}$$

$$\Delta u(x) = x^{\underline{-1}}$$

Summation by Parts

$$\boxed{\sum_{k=0}^{n-1} kH_k} = \sum_{0 \le x < n} xH_x \,\delta x = \textstyle\sum_0^n xH_x \,\delta x$$

$$= \left(\frac{x^{\underline{2}}}{2} H_x - \sum \frac{(x+1)^{\underline{2}}}{2} \cdot x^{\underline{-1}} \delta x \right) \Bigg|_0^n$$

Summation by Parts

Evaluate:

$$\boxed{\frac{(x+1)^{\underline{2}}}{2} \cdot x^{\underline{-1}}} = \frac{1}{2}x(x+1) \cdot \frac{1}{x+1} = \frac{1}{2}x$$

$$= \boxed{\frac{1}{2}x^{\underline{1}}}$$

Summation by Parts

$$\boxed{\sum_{0}^{n-1} kH_k} = \sum_{0 \le x < n} xH_x \delta x$$

$$= \left(\frac{x^{\underline{2}}}{2} H_x - \frac{1}{2} \sum x^{\underline{1}} \delta x \right) \Big|_0^n$$

$$= \left(\frac{x^{\underline{2}}}{2} H_x - \frac{1}{2} \frac{x^{\underline{2}}}{2} \right) \Big|_0^n$$

$$= \frac{x^{\underline{2}}}{2} \left(H_x - \frac{1}{2} \right) \Big|_0^n = \boxed{\frac{n^{\underline{2}}}{2} \left(H_n - \frac{1}{2} \right)}$$

FC Formula

$$\boxed{\sum_{0}^{n-1} kH_k = \frac{n^2}{2}\left(H_n - \frac{1}{2}\right)}$$