cse547, math547 DISCRETE MATHEMATICS

Professor Anita Wasilewska

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LECTURE 9b

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CHAPTER 2 SUMS

- Part 1: Introduction Lecture 5
- Part 2: Sums and Recurrences (1) Lecture 5
- Part 2: Sums and Recurrences (2) Lecture 6
- Part 3: Multiple Sums (1) Lecture 7
- Part 3: Multiple Sums (2) Lecture 8
- Part 3: Multiple Sums (3) General Methods Lecture 8a
- Part 4: Finite and Infinite Calculus (1) Lecture 9a
- Part 4: Finite and Infinite Calculus (2) Lecture 9b
- Part 5: Infinite Sums- Infinite Series Lecture 10

CHAPTER 2 SUMS

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Part 4: Finite and Infinite Calculus (2) - Lecture 9b

Definite Integral and Definite Sum

Infinite Calculus: DEFINITE INTEGRAL

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$$\int_{a}^{b} g(x)dx = f(x)\Big|_{a}^{b} = f(b) - f(a) \quad \text{where} \quad f'(x) = g(x)$$

Finite Calculus: DEFINITE SUM

$$\sum_{a}^{b} g(x)\delta_{x} = f(x)\Big|_{a}^{b} = f(b) - f(a)$$

where $\Delta f(x) = g(x)$

Definite Sum Properties

Definite Sum Properties

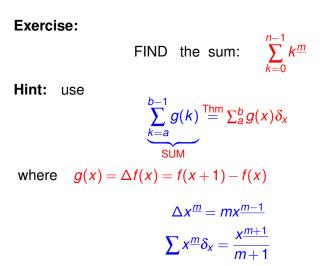
Take b < a and evaluate

(1)
$$\sum_{a}^{b} g(x) \delta_{x} = f(b) - f(a)$$

= $-(f(b) - f(a))$
= $-\sum_{b}^{a} g(x) \delta_{x}$

(2) $\sum_{a}^{b} g(x) \delta_{x} + \sum_{b}^{c} g(x) \delta_{x} = \sum_{a}^{c} g(x) \delta_{x}$ For all $a, b, c \in Z$

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SOLUTION:

$$\sum_{k=0}^{n-1} k^{\underline{m}} \stackrel{(thm)}{=} \sum_{k=0}^{n} x^{\underline{m}} \delta_x$$
$$= \frac{x^{\underline{m+1}}}{\underline{m+1}} \Big|_0^n = \frac{\underline{n}^{\underline{m+1}}}{\underline{m+1}}$$
used:
$$\Delta\left(\frac{x^{\underline{m+1}}}{\underline{m+1}}\right) = \frac{1}{\underline{m+1}} \Delta x^{\underline{m+1}} = \frac{\underline{m+1}}{\underline{m+1}} \cdot x^{\underline{m}} = x^{\underline{m}}$$
We proved

$$\sum_{0 \le k < n} k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1} \Big|_{0}^{n} = \frac{n^{\underline{m+1}}}{m+1} \quad n, m \ge 0$$

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Exercise: Use "integration" to evaluate

 $\sum_{k=0}^{n-1} k = \sum_{0 \le k < n} k$

Observe that

$$k^{1} = k$$
 as $x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$

$$\sum_{k=0}^{n-1} k = \sum_{k=0}^{n-1} k^{\perp} = \sum_{0 \le k < n} k^{\perp}$$

$$\stackrel{\text{Thm}}{=} \sum_{0}^{n} x^{\perp} \delta_{x} = \frac{x^{2}}{2} \Big|_{0}^{n}$$

$$= \frac{n^{2}}{2} = \frac{n(n-1)}{2}$$

because $n^2 = n \cdots (n-2+1) = n(n-1)$ and hence

$$\sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}$$

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Useful Fact

FACT 1
$$k^2 = k^2 + k^1$$

Proof: $x^{\underline{m}} = x(x-1)\cdots(x-m+1)$

$$k^{\underline{2}} = k(k-2+1) = k(k-1)$$

$$k^{\underline{1}} = k$$

$$k^{\underline{2}} + k^{\underline{1}} = k(k-1) + k = k(k-1+1) = k^{2}$$

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 $\sum_{k=0}^{n-1} k^2$

Evaluate

Hint: use Thm and FACT1

$$\begin{split} \sum_{k=0}^{n-1} k^2 &\stackrel{F_1}{=} & \sum_{0 \le k < n} (k^2 + k^1) = \sum_{0 \le k < n} k^2 + \sum_{0 \le k < n} k^1 \\ &\stackrel{\text{Thm}}{=} & \sum_{0}^{n} x^2 \frac{\delta_x}{\delta_x} + \sum_{0}^{n} x^1 \delta_x \\ &= & \frac{x^3}{3} \Big|_{0}^{n} + \frac{x^2}{2} \Big|_{0}^{n} = \frac{n^3}{3} + \frac{n^2}{2} \\ &= & \frac{1}{3} (n(n-1)(n-2)) + \frac{1}{2} (n(n-1)) = \frac{1}{3} n(n-\frac{1}{2})(n-1) \\ & \quad \left(\frac{1}{3} n(n-1)(n-2 + \frac{3}{2})\right) \end{split}$$

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FACT2 $k^3 = k^3 + 3k^2 + k^1$

Prove it as an exercise and use it to evaluate a not trivial sum:

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3$$

$$\begin{split} \sum_{k=1}^{n} k^{3} &\stackrel{F_{1}}{=} \sum_{k=1}^{n} (k^{3} + 3k^{2} + k^{1}) \\ &= \sum_{k=1}^{n+1} k^{3} \delta_{k} + 3 \sum_{k=1}^{n+1} k^{2} \delta_{k} + \sum_{k=1}^{n+1} k^{1} \delta_{k} \\ &= \frac{k^{4}}{4} + 3 \frac{k^{3}}{3} + \frac{k^{2}}{2} \Big|_{1}^{n+1} \end{split}$$

$$\sum_{a \le k < b} k^3 = \frac{k^4}{4} + 3\frac{k^3}{3} + \frac{k^2}{2} \Big|_a^b$$

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Simple Problem

PROVE

$$(x+y)^2 = x^2 + 2x^1y^1 + y^2$$

$$x^{\underline{3}} = x(x-1)(x-2)$$

$$x^{\underline{2}} = x(x-1)$$

$$x^{\underline{1}} = x$$

$$x^{\underline{0}} = 1$$

$$x^{\underline{2}} = \frac{x^{\underline{3}}}{x-2}$$

$$\begin{array}{rcl} x^{-} & - & \overline{x-2} \\ x^{\underline{1}} & = & \frac{x^{\underline{2}}}{x-1} \\ x^{\underline{0}} & = & \frac{x^{\underline{1}}}{x} \end{array}$$

Negative Exponent Falling Powers

Definition of negative exponent falling powers

$$x^{-1} = \frac{1}{x+1}$$
$$x^{-2} = \frac{1}{(x+1)(x+2)}$$
$$x^{-3} = \frac{1}{(x+1)(x+2)(x+3)}$$

General:

$$\boxed{x^{\underline{-m}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}} \qquad m > 0$$

Problems

Prove:

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$$

Prove: for
$$m < 0$$

$$\Delta x^{\underline{m}} = m x^{\underline{m-1}}$$

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Example:

$$\boxed{\Delta x^{-2}} = \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)}$$
$$= \frac{(x+1) - (x+3)}{(x+1)(x+2)(x+3)}$$
$$= \boxed{-2x^{-3}}$$

Fact:

$$\sum_{a}^{b} x^{\underline{m}} \delta x = \frac{x^{\underline{m+1}}}{\underline{m+1}} \Big|_{a}^{b} \quad \text{all } m \neq -1$$

What about case m = -1?

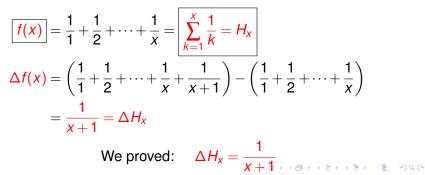
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Case m = -1:

Infinite Integral $\int_{a}^{b} x^{-1} dx = \int_{a}^{b} \frac{1}{x} dx = \ln|x| \Big|_{a}^{b}$

We want to have a **finite analog**: $x = \frac{1}{x+1} \quad | \quad \Delta f = f(x+1) - f(x)$





Case
$$m = -1$$

$$\sum_{a}^{b} x^{-1} \delta x = \sum_{a}^{b} \frac{1}{x+1} \delta x = H_{x} |_{a}^{b}$$

We prove (Chapter 9) that for large x

$$H_x - \ln x \approx 0.577 + \frac{1}{2x}$$

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 $H_x \sim \ln x$ as do \int_a^b and \sum_a^b

Falling Powers Theorem

Theorem: Sums of falling powers

$$\sum_{a}^{b} x^{\underline{m}} \delta x = \begin{cases} \frac{x^{\underline{m+1}}}{\underline{m+1}} \Big|_{a}^{b} & m \neq -1 \\ H_{x} \Big|_{a}^{b} & m = -1 \end{cases}$$

all $m \in \mathbb{Z}$

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and
$$\int_a^b \frac{1}{x} dx = \ln|x||_a^b$$
 is similar to $\sum_a^b x^{-1} = H_x|_a^b$

More Similarities

More Similarities

We know $(e^x)' = e^x$, i.e. $De^x = e^x$ Df = f when $f(x) = e^x$

Question

Can we have a function f that has a **similar** property for Δ ?

i.e. a function f such that $\Delta f(x) = f(x)$ $\Delta f(x) = f(x+1) - f(x) = f(x)$ Answer: Any f such that: f(x+1) = 2f(x) Recurrence!

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Example of solution:

Find

$$f(x) = 2^{x}$$

$$f(x+1) - f(x) = 2^{x+1} - 2^{x}$$

$$= 2 \cdot 2^{x} - 2^{x}$$

$$= 2^{x} = f(x)$$
We proved: $\Delta(2^{x}) = 2^{x}$ $(e^{x})' = e^{x}$
a formula for Δf , where $f(x) = c^{x}$ for $c \in N^{+}$

$$\Delta(c^{x}) = c^{x+1} - c^{x} = c \cdot c^{x} - c^{x}$$

$$= c^{x}(c-1)$$
We proved: $\Delta(c^{x}) = c^{x}(c-1)$

Difference

Difference:

$$\Delta(c^{x}) = (c-1)c^{x} \qquad c \in N^{+} \qquad \text{``derivative''}$$

$$\sum_{a}^{b} c^{x} \delta x = \frac{c^{x}}{c-1} \Big|_{a}^{b} \qquad c \neq 1$$

"antiderivative" **anti-difference**

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Geometric Progression

We now prove: Theorem: Geometric Progression

$$\frac{\sum_{a \le k < b} c^k}{\left| = \sum_a^b c^x \delta x \right|} = \frac{c^x}{c-1} \Big|_a^b = \frac{c^b - c^a}{c-1} \qquad c \neq 1$$

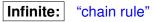
General Formula for Geometric Progression

$$\boxed{\sum_{\substack{a \leq k < b}} c^k = \frac{c^b - c^a}{c - 1}} c^k$$

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Chain Rule



 $Df(g(x)) = Df \cdot Dg(x)$

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no such rule

Can't relate $\Delta f(g(x))$ to $\Delta g(x)$

Integration by Parts

Infinite D(uv) = uDv + vDu

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

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Can we have an **analog** for Δ ?

Integration by Parts

Can we have

and
$$\Delta(uv) = u\Delta v + v\Delta u$$
$$\sum u \,\delta v = uv - \sum v \,\delta u?$$
change here

Not exactly, but close!

Evaluate:

$$\Delta(u(x)v(x)) = u(x+1)v(x+1) - u(x)v(x)$$

= $u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x)$
= $u(x)v(x+1) - u(x)v(x) + u(x+1)v(x+1) - u(x)v(x+1)$
= $u(x)(v(x+1) - v(x)) + v(x+1)(u(x+1) - u(x))$
= $u(x)\Delta v(x) + v(x+1) \Delta u(x)$

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We define a shift operator:

$$Ev(x) = v(x+1)$$

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We proved

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

Summation by parts

$$\sum u\,\delta v = uv - \sum Ev\delta u$$

$$\sum_{a}^{b} u \,\delta v = u v \big|_{a}^{b} - \sum_{a}^{b} E v \,\delta u$$

Integration

$$\int x e^x dx = x e^x - \int 1 \cdot e^x dx = \boxed{e^x (x-1) + C}$$

Summation

$$\boxed{\sum x 2^{x} \delta x} = x 2^{x} - \sum 1 \cdot 2^{x+1} \delta x = \boxed{x 2^{x} - 2^{x+1} + C(x)}$$

for $C(x) = C(x+1)$

Evaluate

$$u(x) = x, \quad v(x) = 2^{x}, \quad Ev(x) = 2^{x+1}$$

 $\Delta u(x) = 1, \quad \Delta v(x) = 2^{x}$
Fact: $\Delta (2^{x+1}) = 2^{x+1}$

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In particular, evaluate:

$$\sum_{k=0}^{n} k2^{k} = 1 \cdot 2^{1} + 2 \cdot 2^{2} + \dots + n \cdot 2^{n}$$

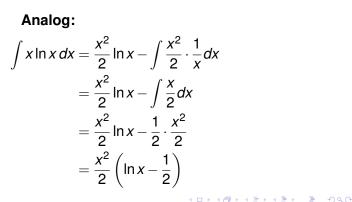
$$\sum_{k=0}^{n} k2^{k} = \sum_{0}^{n+1} x2^{x} \delta x = (x2^{x} - 2^{x+1})|_{0}^{n+1}$$
$$= ((n+1)2^{n+1} - 2^{n+2}) - (0 \cdot 2^{0} - 2)$$
$$= (n+1)2^{n+1} - 2 \cdot 2^{n+1} + 2$$
$$= (n+1-2)2^{n+1} + 2 = (n-1)2^{n+1} + 2$$

$$\sum_{k=0}^{n} k2^{k} = (n-1)2^{n+1} + 2$$

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Use finite calculus to evaluate:





Use
$$\sum u \delta v = uv - \sum Ev \delta u$$

 $Ev(x) = v(x+1)$

$$\sum_{0 \le x,n} x H_x \delta x = uv - \sum Ev \, \delta u$$

$$\Delta v(x) = x = x^{1}$$

$$v(x) = \frac{x^{2}}{2}, \quad v(x+1) = \frac{(x+1)^{2}}{2}$$

$$\Delta u(x) = \Delta H_{x} = x^{-1}$$

$$v(x) = \frac{(x+1)^{2}}{2}$$

$$Ev(x) = \frac{(x+1)^{2}}{2}$$

$$u(x)=H_x$$

 $\Delta u(x) = x^{-1}$

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$$\boxed{\sum_{k=0}^{n-1} kH_k} = \sum_{0 \le x < n} xH_x \delta x = \sum_0^n xH_x \delta x$$
$$= \left(\frac{x^2}{2}H_x - \sum \frac{(x+1)^2}{2} \cdot x^{-1} \delta x\right)\Big|_0^n$$

Evaluate:

$$\frac{\frac{(x+1)^2}{2} \cdot x^{-1}}{2} = \frac{1}{2}x(x+1) \cdot \frac{1}{x+1} = \frac{1}{2}x$$
$$= \frac{1}{2}x^{1}$$

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$$\boxed{\sum_{0}^{n-1} kH_k} = \sum_{0 \le x < n} xH_x \delta x$$
$$= \left(\frac{x^2}{2}H_x - \frac{1}{2}\sum_{x \ge 1} x \delta x\right)\Big|_0^n$$
$$= \left(\frac{x^2}{2}H_x - \frac{1}{2}\frac{x^2}{2}\right)\Big|_0^n$$
$$= \frac{x^2}{2}\left(H_x - \frac{1}{2}\right)\Big|_0^n = \boxed{\frac{n^2}{2}\left(H_n - \frac{1}{2}\right)}$$

FC Formula

$$\sum_{0}^{n-1} kH_k = \frac{n^2}{2} \left(H_n - \frac{1}{2} \right)$$

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