cse547
DISCRETE MATHEMATICS

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Lecture 16

REVIEW for FINAL

Classical DISCRETE MATHEMATICS Problems

Some Discrete Mathematics Problems

PART 1: FINITE and INFINITE SETS

Finite and Infinite Sets

Definition 1

A set A is FINITE if and only if there is a natural number $n \in N$ and there is a $1-1$ function f that maps the set $\{1, 2, ...n\}$ onto A, i.e.

$$f: A \overset{1-1, onto}{\longrightarrow} B$$

Definition 2

A set A is INFINITE if and only if it is NOT FINITE

Problem 1

Use the above definitions 1, 2 to prove the following

Fact 1

A set A is INFINITE if and only if it contains a countably infinite subset, i.e. one can define a $1-1$ sequence $\{a_n\}_{n \in N}$ of some elements of A

Proof

Part 1 Proof of the Implication

If A is infinite, then we can define a 1-1 sequence of elements of A

Fact 1 Proof

Let A be infinite

We define the 1-1 sequence of elements of A

$$a_1,\ a_2, \dots, \dots, \dots a_n,\ \dots$$

as follows

Observe that $A \neq \emptyset$, because if $A = \emptyset$, the set A would be finite. Contradiction

So there is an element of $a \in A$ and we define

$$a_1 = a$$

Consider now a set $A - \{a_1\} = A_1$. $A_1 \neq \emptyset$ because if $A_1 = \emptyset$, then $A - \{a_1\} = \emptyset$ and A would be finite. Contradiction

So there is an element $a_2 \in A - \{a_1\}$ and $a_1 \neq a_2$ and we define

$$a_1,\ a_2$$

Fact 1 Proof

Assume that we defined a set

$$A_n = A - \{a_1, \dots, a_n\}$$

The set $A_n \neq \emptyset$ because if $A - \{a_1, \dots, a_n\} = \emptyset$, then A is finite. Contradiction

So there is an element

$$a_{n+1} \in A - \{a_1, \dots, a_n\} \text{ and } a_{n+1} \neq a_n \dots \neq a_1$$

By mathematical induction, we have defined a 1-1 sequence

$$a_1, \; a_2, \dots, \dots, ... \; a_n, \; \dots$$

of elements of A

This ends the proof the Part 1

Fact 1 Proof

Part 2 Proof of the Implication

If the set A contains a 1-1 sequence
$a_1,\ a_2, \ldots, \ldots, \ldots\ a_n,\ \ldots$, *then A is INFINITE*

Assume A is NOT INFINITE; i.e by the Definition 1 A is finite. Every subset of finite set is finite, so we can't have a 1-1 infinite sequence of elements of A. Contradiction

Finite and Infinite Sets - Problem 2

Problem 2

Use the **Fact 1** from **Problem 1** to prove the following

Dedekind Theorem

For any set A,

A is INFINITE if and only if there is a proper subset B of the set A, such that $|A| = |B|$

Dedekind Theorem is sometimes used as a definition of the infinite set

Dedekind Theorem Proof

Dedekind Theorem

For any set A,

A is INFINITE if and only if there is a proper subset B of the set A, such that $|A| = |B|$

Proof

Part 1 Proof of the Implication

If A is an infinite set, then there is a set B and there is a function f such that

$$B \subset A \quad \text{and} \quad f: \; A \stackrel{1-1, onto}{\longrightarrow} B$$

Dedekind Theorem Proof

Let the set A be infinite

By the **Fact 1**, we have a 1-1 sequence $a_1, a_2, \ldots, a_n, \ldots$ of elements of A

We take $B = A - \{a_1\}$. Obviously $B \subset A$ and we define the function $f: A \xrightarrow{1-1, onto} B$ as follows

$$f(a_1) = a_2, \quad f(a_2) = a_3 \quad \ldots \quad f(a_n) = a_{n+1}$$

$$f(a) = a \text{ for all other } a \in A$$

Obviously, f is 1-1, onto

Observe: we have many of other choices of the set B

Dedekind Theorem Proof

Part 2 Proof of the Implication

If there is a proper subset B of the set A, such that $|A| = |B|$, then the set A is INFINITE

Assume that we have $B \subset A$ and the function f, such that

$$f : \; A \overset{1-1, onto}{\longrightarrow} B$$

We use **Fact 1** to show that is infinite; i.e we do it by constructing a 1-1 sequence $a_1 \dots a_n, \dots$ of elements of A

We do it as follows

We know that $B \subset A$, so $A - B \neq \emptyset$ and there is $b \in A - B$

This is our first element of the sequence $a_1 \dots a_n, \dots$

Dedekind Theorem Proof

Observe that $f: A \overset{1-1, onto}{\longrightarrow} B$, so $f(b) \in B$ and $b \in A - B$, hence $f(b) \neq b$ and we take $f(b)$ is our second element of the sequence

We have now,

$$a_1 = b, \ a_2 = f(b)$$

and $f(b) \neq b, \ b \in A - B, f(b) \in B$

Take now a new element $ff(b)$

As f is 1-1 and $f(b) \neq b$, we get $ff(b) \neq f(b) \neq b$ and

we defined a one- one finite sequence

$$a_1 = b, \ \ a_2 = f(b), \ \ a_3 = ff(b)$$

We denote $ff(b) = f^2(b)$

Dedekind Theorem Proof

We continue the construction by mathematical induction

Assume that we have constructed a 1-1 finite sequence

$$a_1 = b, \;\; a_2 = f(b), \;\; a_3 = f^2(b)f^3(b), \;\; \ldots, \;\; f^n(b)$$

Observe that

$$ff^n(b) = f^{n+1}(b) \neq f^n(b)$$ as the function f is 1-1

By mathematical induction, we have that the sequence

$$\{f^n(b)\}_{n \in N}$$

is a 1-1 sequence of elements of A and hence by **Fact 1** A is **infinite**

Problem 3

Use technique from the proof of **Dedekind Theorem** to prove the following

Fact 2

For any infinite set A and its finite subset B, $|A| = |A - B|$

Proof

A is infinite, then by **Fact 1** there is a 1-1 sequence $a_1,\ a_2,\ \dots,\ \ a_n,\ \dots$ of elements of A

Let $|B| = k$

We choose k 1-1 sequences $\{c_n^k\}_{n \in N}$ of the sequence $\{a_n\}_{n \in N}$, such that

$$c_n^j \neq c_n^i \ \text{ for all } \ j \neq i,\ 1 \leq i, j \leq k \ \text{ and } \ n \in N$$

Fact 2 Proof

Let $B = \{b_1, \dots, b_k\}$

We construct a function $f: A \overset{1-1, onto}{\longrightarrow} A - \{b_1, \dots, b_k\}$ as follows

$$f(b_1) = c_1^1, \qquad f(c_1^1) = c_2^1, \dots, f(c_n^1) = c_{n+1}^1$$

$$f(b_2) = c_1^2, \qquad f(c_1^2) = c_2^2, \dots, f(c_n^2) = c_{n+1}^2$$

$$\vdots$$

$$f(b_k) = c_1^k, \qquad f(c_1^k) = c_2^k, \dots, f(c_n^k) = c_{n+1}^k$$

$$f(a) = a \quad \text{for all} \quad a \in A - B$$

As all sequences $\{C_n^m\}_{n \in N, m=1,\dots,k}$ are 1-1, and different, the function f is 1-1 and obviously ONTO $A - B$

Problem 4

Use technique from the proof of **Dedekind Theorem** to prove that the interval $[a, b],\ a < b$ of real numbers is infinite and

$$|[a, b]| = |(a, b)|$$

Solution

Use construction presented in the proof of the **Fact 2** to construct a function

$$f : \quad [a, b] \xrightarrow{1-1, onto} (a, b)$$

Problem 5

Prove the following

Fact 3

For any or any cardinal numbers $\mathcal{M}, \mathcal{N}, \mathcal{K}$,

$$1. \quad \mathcal{N} \leq \mathcal{N}$$

$$2. \quad \text{If} \quad \mathcal{N} \leq \mathcal{M} \quad \text{and} \quad \mathcal{M} \leq \mathcal{K}, \quad \text{then} \quad \mathcal{N} \leq \mathcal{K}$$

Solution

1. $\mathcal{N} \leq \mathcal{N}$ means that for any set A, we have that $|A| \leq |A|$

It is established for example, by taking $f(a) = a$, for all $a \in A$, as obviously

$$f : \quad A \xrightarrow{1-1} A$$

Problem 4 Solution

2. If $\mathcal{N} \leq \mathcal{M}$ and $\mathcal{M} \leq \mathcal{K}$, then $\mathcal{N} \leq \mathcal{K}$

Solution

We have sets A, B, C, such that $|A| = \mathcal{N}$, $|B| = \mathcal{M}$ and $|C| = \mathcal{K}$ and we assume that there are functions f and g, such that

$$f: \quad A \stackrel{1-1}{\longrightarrow} B \quad \text{and} \quad g: B \stackrel{1-1}{\longrightarrow} C$$

We have to construct a function h, such that

$$h: \quad A \stackrel{1-1}{\longrightarrow} C$$

We take as h a composition of f and g, i.e. we put for all $a \in A$, $h(a) = g(f(a))$ and h is obviously 1- 1

Problem 5

Use Mathematical Induction to prove the following property of finite posets

Property 1 Every non-empty finite poset has at least one maximal element

Proof

Let $(A, \leq)$ be a finite, not empty poset such that A has n-elements, i.e. $|A| = n$

We carry the Mathematical Induction over $n \in N - \{0\}$

Reminder: an element $a_o \in A$ is a maximal element in a poset $(A, \leq)$ if and only if

$$\neg\exists_{a \in A}(a_0 \neq a \cap a_0 \leq a)$$

Inductive Proof

Base case: $n = 1$, so $A = \{a\}$ and a is maximal (and minimal, and smallest, and largest) in the poset $(\{a\}, \leq)$

Inductive step: Assume that any set A such that $|A| = n$ has a maximal element;

Denote by a_0 the maximal element in $(A, \leq)$

Let B be a set with $n + 1$ elements; i.e. we can write B as

$B = A \cup \{b_0\}$ for $b_0 \notin A$, for some A with n elements

Inductive Proof

By **Inductive Assumption** the poset $(A, \leq)$ has a maximal element a_0

To show that $(B, \leq)$ has a maximal element we need to consider 3 cases.

1. $b_0 \leq a_0$; in this case a_0 is also a maximal element in $(B, \leq)$

2. $a_0 \leq b_0$; in this case b_0 is a new maximal in $(B, \leq)$

3. a_0, b_0 are not compatible; in this case a_0 remains maximal in $(B, \leq)$

By Mathematical Induction we have proved that

$\forall_{n \in \in N-\{0\}}(|A| = n \Rightarrow$ A has a maximal element)

Problem 6

Definition

Let D be a set, let $n \geq 0$ and

let $R \subseteq D^{n+1}$ be a $(n+1)$-ary relation on D

Then the subset B of D is said to be **closed under** R

if $b_{n+1} \in B$ whenever $(b_1, \dots, b_n, b_{n+1}) \in R$

Any property of the form " the set B is closed under relations $R_1, R_2, \dots, R_m$" is called a **Closure Property** of B

CP Theorem

Prove the following **Closure Property Theorem**

CP Theorem

Let P be a closure property defined by relations on a set D, and let $A \subseteq D$

Then there is a **unique minimal** set B such that $B \subseteq A$ and B has property P

Proof Consider the set if all subsets of D that are **closed** under relations $R_1, R_2, \ldots, R_m$ and that have A as a subset

We call this set $\mathcal{S}$

CP Theorem Proof

Consider now

$$\mathcal{S} = \{X \in 2^D : \ A \subseteq X \text{ and } X \text{ is closed under } R_1, R_2, \dots, R_m\}$$

We need to show that the poset $\mathbf{S} = (\mathcal{S}, \subseteq)$ has a **unique minimal** element B.

Observe that $\mathcal{S} \neq \emptyset$ as $D \subseteq \mathcal{S}$ and D is trivially closed under $R_1, R_2, \dots, R_m$ and by definition $A \subseteq D$.

Consider then the set B which is the intersection of all sets in $\mathcal{S}$, i.e.

$$B = \bigcap \mathcal{S}$$

Obviously $A \subseteq B$ and we have to show now that B is closed under all R_i

CP Theorem Proof

Suppose that $a_1, a_2, \ldots a_{n-1} \in B$, and

$a_1, a_2, \ \ldots \ a_{n-1}, \ a_n \in R_i$

Since B is the intersection of all sets in $\mathcal{S}$, we have that

$a_1, a_2, \ldots a_{n-1} \in X$, for all $X \in \mathcal{S}$

But all sets in $\mathcal{S}$ are closed under all R_i, they also contain a_n

Therefore $a_n \in B$ and hence B is closed under all R_i

Moreover, B is **minimal**, because there can be **no proper** subset C of B, such that $A \subseteq C$ and C is closed under all R_i

Because then C would be a member of $\mathcal{S}$ and thus C would include B