cse547
DISCRETE MATHEMATICS
Review for FINAL: CONCRETE MATHEMATICS

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CHAPTER 2
PART 5: INFINITE SUMS (SERIES)

Infinite Series
D

Must Know STATEMENTS- do not need to PROVE the Theorems

Definition

If the limit $\lim_{n\to\infty} S_n$ **exists** and **is finite**, i.e.

$$\lim_{n\to\infty} S_n = S,$$

then we say that the infinite sum $\Sigma_{n=1}^{\infty} a_n$ **converges** to S and we write

$$\Sigma_{n=1}^{\infty} a_n = \lim_{n\to\infty} \Sigma_{k=1}^{n} a_k = S,$$

otherwise the infinite sum **diverges**

Example

Show

The infinite sum $\Sigma_{n=1}^{\infty} (-1)^n$ **diverges**

The infinite sum $\Sigma_{n=0}^{\infty} \frac{1}{(k+1)(k+2)}$ **converges** to 1

Example

Example

The infinite sum $\Sigma_{n=0}^{\infty} (-1)^n$ **diverges**

Proof

We use the Perturbation Method

$$S_n + a_{n+1} = a_0 + \sum_{k=0}^{n} a_{k+1}$$

to eveluate

$$S_n = \Sigma_{k=0}^{n} (-1)^k = \frac{1+(-1)^n}{2} = \frac{1}{2} + \frac{(-1)^n}{2}$$

and we prove that

$$\lim_{n\to\infty} \left(\frac{1}{2} + \frac{(-1)^n}{2}\right)$$ **does not exist**

Example

The **infinite sum** $\Sigma_{n=0}^{\infty} \frac{1}{(k+1)(k+2)}$ **converges** to 1; i.e.

$$\Sigma_{n=0}^{\infty} \frac{1}{(k+1)(k+2)} = 1$$

Proof: first we evaluate $S_n = \Sigma_{k=0}^{n} \frac{1}{(k+1)(k+2)}$ as follows

$$S_n = \Sigma_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \Sigma_{k=0}^{n} k^{\underline{-2}} = \Sigma_{k=0}^{n+1} k^{\underline{-2}} \, \delta k$$

$$= -\frac{1}{k+1}\Big|_{0}^{n+1} = -\frac{1}{n+2} + 1$$

and

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} -\frac{1}{n+2} + 1 = 1$$

Theorem

Theorem

If the infinite sum $\Sigma_{n=1}^{\infty} a_n$ **converges**, then $\lim_{n\to\infty} a_n = 0$

Observe that this is equivalent to

If $\lim_{n\to\infty} a_n \neq 0$ then $\Sigma_{n=1}^{\infty} a_n$ diverges

The **reverse** statement

If $\lim_{n\to\infty} a_n = 0$, then $\Sigma_{n=1}^{\infty} a_n$ converges is not always true

The **infinite harmonic sum** $H = \Sigma_{n=1}^{\infty} \frac{1}{n}$ **diverges** to ∞ even if $\lim_{n\to\infty} \frac{1}{n} = 0$

Theorem

Theorem (D'Alambert's Criterium)

If $a_n \geq 0$ and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$

then the series $\sum\limits_{n=1}^{\infty} a_n$ **converges**

Theorem (Cauchy's Criterium)

If $a_n \geq 0$ and $\lim\limits_{n\to\infty} \sqrt[n]{a_n} < 1$

then the series $\sum\limits_{n=1}^{\infty} a_n$ **converges**

Theorems

Theorem (Divergence Criteria)

If $a_n \geq 0$ and $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$

then the series $\sum_{n=1}^{\infty} a_n$ **diverges**

Convergence/Divergence

Remark

It can happen that for a certain infinite sum $\sum_{n=1}^{\infty} a_n$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 = \lim_{n\to\infty} \sqrt[n]{a_n}$$

In this case our Divergence Criteria **do not decide** whether the infinite sum **converges** or **diverges**

We say in this case that that the infinite sum does not react on the criteria

There are other, stronger criteria for **convergence** and **divergence**

Examples

Example

The Harmonic series $H = \sum_{n=1}^{\infty} \frac{1}{n}$ **does not react** on D'Alambert's Criterium

Proof: Consider

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})} = 1$$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ we say , that the **Harmonic series**

$$H = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not react on D'Alambert's criterium

Examples

Example

The series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ **does not react** on D'Alambert's Criterium (

Proof:

Consider, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+4n+1} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{4}{n}+\frac{4}{n^2}} = 1$$

Since, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ we say , that the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

does not react on D'Alambert's criterium

Example 1

$$\sum_{n=1}^{\infty} \frac{c^n}{n!} \quad converges \quad for \quad c > 0$$

$$HINT: \; Use \; D'Alembert$$

Proof:

$$\frac{a_{n+1}}{a_n} = \frac{c^{n+1}}{c^n} \frac{n!}{(n+1)!}$$

$$= \frac{c}{n+1}$$

Example

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{c}{n+1}$$

$$= 0 < 1$$

By D'Alembert's Criterium

$$\sum_{n=1}^{\infty} \frac{c^n}{n!} \quad \textbf{converges}$$

Example

Example

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \qquad \textit{converges}$$

Proof:

$$a_n = \frac{n!}{n^n}$$

$$a_{n+1} = \frac{n!(n+1)}{(n+1)^{n+1}}$$

$$\frac{a_n + 1}{a_n} = \frac{n! \;\; n^{(n+1)}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= (n+1) \cdot \frac{n^n}{(n+1)^{n+1}}$$

Example

$$
\begin{aligned}
(n+1)^{n+1} &= (n+1)^n \;(n+1) \\
\frac{a_n+1}{a_n} &= \frac{(n+1)\;\; n^n}{(n+1)^n \quad (n+1)} \\
&= \left(\frac{n}{n+1}\right)^n \\
&= \frac{1}{(1+\frac{1}{n})^n}
\end{aligned}
$$

Example

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n}$$

$$= \frac{1}{e} \quad < 1$$

By D'Alembert's Criterium the series,

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \quad \textbf{converges}$$

Exercise

Exercise

Prove that

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0 \qquad \textit{for} \quad c > 0$$

Solution:

We have proved in **Example**

$$\sum_{n=1}^{\infty} \frac{c^n}{n!} \quad \text{converges} \quad \text{for } c > 0$$

Exercise

Theorem says:

$$\text{IF} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \text{THEN} \quad \lim_{n\to\infty} a_n = 0$$

Hence by **Example** and **Theorem** we have proved that

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0 \quad \textit{for} \quad c > 0$$

Observe that we have also proved that $n!$ grows faster than c^n

Homework Problem 1

Part 1 Prove that

$$\sum_{k=2}^{n} \frac{(-1)^k}{2k-1} = -\sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1}$$

Part 2 Use partial fractions and Part 1 result (must use it!) to evaluate the sum

$$S = \sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2-1)}$$

Homework Problem 2

Show that the nth element of the sequence:

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, ...$$

is $\lfloor \sqrt{2n} + \frac{1}{2} \rfloor$

Hint: Let P(x) represent the position of the last occurrence of x in the sequence. Use the fact that $P(x) = \frac{x(x+1)}{2}$

Let the nth element be m. You need to **find** m

CHAPTER 3
INTEGER FUNCTIONS

Here are the **proofs** in course material you need to know for **Final**

Plus, of course the regular Homework Problems

PART1: Floors and Ceilings

Prove the following

Fact 3

For any $x, y \in R$

$$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \quad \text{when} \quad 0 \leq \{x\} + \{y\} < 1$$

and

$$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1 \quad \text{when} \quad 1 \leq \{x\} + \{y\} < 2$$

Fact 5

For any $x \in R,\ x \geq 0$ the following property holds

$$\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$$

Prove the Combined Domains Property

Property 4

$$\sum_{Q(k)\cup R(k)} a_k = \sum_{Q(k)} a_k + \sum_{R(k)} a_k - \sum_{Q(k)\cap R(k)} a_k$$

where, as before,

$k \in K$ and $K = K_1 \times K_2 \cdots \times K_i$ for $1 \leq i \leq n$

and the above formula represents single (i =1)

and multiple $(i > 1)$ sums

Spectrum

Definition

For any $\alpha \in R$ we define a SPECTRUM of α as

$$Spec(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor \cdots\}$$

$$Spec(\sqrt{2}) = \{1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, \cdots\}$$

$$Spec(2 + \sqrt{2}) == \{3, 6, 10, 13, 17, 20, \cdots\}$$

Spectrum Partition Theorem

Spectrum Partition Theorem

1. $Spec(\sqrt{2}) \neq \emptyset$ and $Spec(2+\sqrt{2}) \neq \emptyset$
2. $Spec(\sqrt{2}) \cap Spec(2+\sqrt{2}) = \emptyset$
3. $Spec(\sqrt{2}) \cup Spec(2+\sqrt{2}) = N - \{0\}$

Finite Partition Theorem

First, we define certain **finite subsets** $A_n,\ B_n$ of $Spec(\sqrt{2})$ and $Spec(2+\sqrt{2})$, respectively

Definition

$$A_n = \{m \in Spec(\sqrt{2}) : \quad m \leq n\}$$

$$B_n = \{m \in Spec(2+\sqrt{2}) \quad m \leq n\}$$

Remember

A_n and B_n are subsets of $\{1, 2, \dots n\}$ for $n \in N - \{0\}$

Finite Partition Theorem

Given sets

$A_n = \{m \in Spec(\sqrt{2}) : \quad m \leq n\}$

$B_n = \{m \in Spec(2+\sqrt{2}) : \quad m \leq n\}$

Finite Spectrum Partition Theorem

1. $A_n \neq \emptyset$ and $B_n \neq \emptyset$

2. $A_n \cap B_n = \emptyset$

3. $A_n \cup B_n = \{1, 2, \ldots n\}$

Counting Elements

Before trying to prove the **Finite Fact** we first look for a closed formula to count the number of elements in subsets of a finite size of any spectrum

Given a spectrum $Spec(\alpha)$

Denote by $N(\alpha, n)$ the number of elements in the $Spec(\alpha)$ that are $\leq n$, i.e.

$$N(\alpha, n) = |\{m \in Spec(\alpha): \quad m \leq n\}|$$

Spectrum Partitions

1. Justify that

$$N(\alpha, n) = \sum_{k>0} \left[k < \frac{n+1}{\alpha} \right]$$

2. Write a detailed proof of

$$N(\alpha, n) = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1$$

3. Write a detailed proof of **Finite Fact**

$$|A_n| + |B_n| = n \qquad \text{for any } \; n \in N - \{0\}$$

Spectrum Partitions

Finite Spectrum Partition Theorem

1. $A_n \neq \emptyset$ and $B_n \neq \emptyset$
2. $A_n \cap B_n = \emptyset$
3. $A_n \cup B_n = \{1, 2, \ldots n\}$

Spectrum Partitions

Prove - use your favorite proof out of the two I have provided

Spectrum Partition Theorem

1. $Spec(\sqrt{2}) \neq \emptyset$ and $Spec(2+\sqrt{2}) \neq \emptyset$
2. $Spec(\sqrt{2}) \cap Spec(2+\sqrt{2}) = \emptyset$
3. $Spec(\sqrt{2}) \cup Spec(2+\sqrt{2}) = N - \{0\}$

Generalization

Prove

General Spectrum Partition Theorem

Let $\alpha > 0,\ \beta > 0,\quad \alpha,\ \beta \in R - Q$ be such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

Then the sets

$$A = \{\lfloor n\alpha \rfloor : \quad n \in N - \{0\}\} = Spec(\alpha)$$

$$B = \{\lfloor n\beta \rfloor : \quad n \in N - \{0\}\} = Spec(\beta)$$

form a **partition** of $Z^+ = N - \{0\}$, i.e.

1. $A \neq \emptyset$ and $B \neq \emptyset$

2. $A \cap B = \emptyset$

3. $A \cup B = Z^+$

PART3: Sums

Write detailed evaluation of

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor$$

Hint: use

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor = \sum_{0 \le k < n} \quad \sum_{m \ge 0,\ m = \lfloor \sqrt{k} \rfloor} m$$

Chapter 4 Material in the Lecture 12

P1.

JUSTIFY correctness of the following example and be ready to do similar problems upwards or downwards

Represent 19151 in a system with base 12

Example

$$19151 = 1595 \cdot 12 + 11$$

$$1595 = 132 \cdot 12 + 11$$

$$132 = 11 \cdot 12 + 0$$

$$a_0 = 11, \quad a_1 = 11, \quad a_2 = 0, \quad a_3 = 11$$

So we get

$$19151 = (11, 0, 11, 11)_{12}$$

Chapter 4

P2. Write a **proof** of Step 1 and Step 2 of the

Proof of Correctness of **Euclid Algorithm**

P3. Use Euclid Algorithm to prove the following

Fact 1 When a product ac of two natural numbers is divisible by a number b that is **relatively prime** to a, the factor c must be divisible by b

P4. Use Euclid Algorithm to prove the following

Fact 2

$$gcd(ka, kb) = k \cdot gcd(a, b)$$

Chapter 4

P5.

Prove:

Any common multiple of a and b is **divisible** by lcm(a,b)

P6.

Prove the following

$$\forall_{p,q_1 q_2 \dots q_n \in P} \left(p \mid \prod_{k=1}^{n} q_k \Rightarrow \exists_{1 \leq i \leq n} \left(p = q_i\right)\right)$$

P7.

Write down a formal formulation (in all details) of the **Main Factorization Theorem** and its General Form

P8.

Prove that the representation given by Main Factorization Theorem is unique

Explain why and show that $18 =< 1, 2 >$

Prove

$$k = gcd(m, n) \quad \text{if and only if} \quad k_p = min\{m_p, n_p\}$$

$$k = lcd(m, n) \quad \text{if and only if} \quad k_p = max\{m_p, n_p\}$$

P9.

Let

$$m = 2^0 \cdot 3^3 \cdot 5^2 \cdot 7^0 \qquad n = 2^0 \cdot 3^1 \cdot 5^0 \cdot 7^3$$

Evaluate gcd(m, n) and k = lcd(m, n)

Exercises

P10.

Prove

Theorem

For any $a, b \in Z^+$ such that lcm(a,b) and gcd(a, b) exist

$$lcm(a, b) \cdot gcd(a, b) = ab$$

Chapter 5

Study Lectures and Homework Problems

Use them to solve the following

Problem 1 Prove that

$$\binom{x}{m}\binom{m}{k} = \binom{x}{k}\binom{x-k}{m-k}$$

holds for all $m, k \in Z$ and $x \in R$

Consider all cases and **Polynomial argument**

Problem 2 Prove the Hexagon property for $n, k \in N$

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}$$

Chapter 5

Problem 3 Evaluate

$$\sum_k \binom{n}{k}^3 (-1)^k$$

use the formula

$$\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \frac{(a+b+c)!}{a!b!c!}$$