Hodge Decomposition

David Gu[1,2]

[1]Computer Science Department
Stony Brook University
Yau Mathematical Sciences Center
Tsinghua University

Tsinghua University

Hodge Decomposition

Philosophy

In each cohomnologous class, there exists a unique harmonic form, which is the smoothest one in the whole class.
Each 1-form is dual to a vector field, the harmonic 1-form corresponds to the vector field, which is with zero curl and zero divergence.

Poincaré Dual

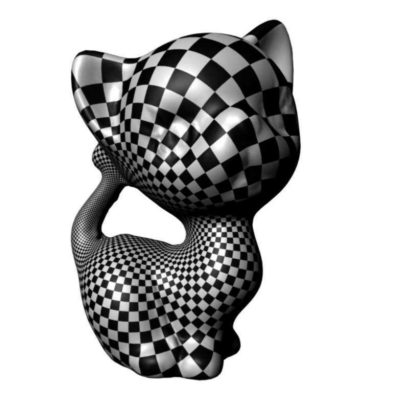

Figure: Holomorphic 1-form

Poincaré Dual

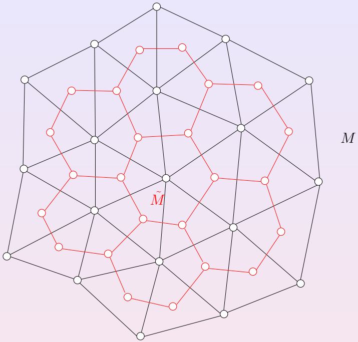

Figure: Poincaré Dual

Each face in M corresponds to a vertex in $\tilde{M}$, each vertex in M is dual to a face in $\tilde{M}$, each edge in M corresponds to an edge in $\tilde{M}$.

Poincaré Dual

Theorem (Poincaré Dual)

If M is a n-dimensional closed manifold, then

$$H_k(M,\mathbb{Z}) \cong H_{n-k}(M,\mathbb{Z}), H^k(M,\mathbb{Z}) \cong H^{n-k}(M,\mathbb{Z}).$$

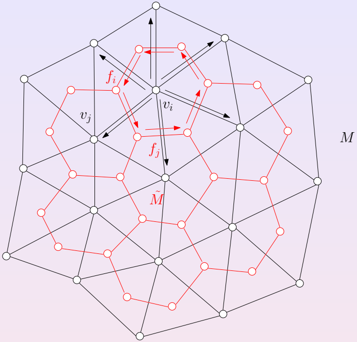

Figure: Hodge Star Operator

Suppose a vertex $v_i \in M$ is dual to a face $\tilde{v}_i$, $\sigma \in C^0(M, \mathbb{R})$, then ${}^*\sigma \in C^2(\tilde{M}, \mathbb{R})$, such that

$$\sigma(v_i) = {}^*\sigma(\tilde{v}_i).$$

Hodge Star

Suppose a face $f_i \in M$ is dual to a vertex $\tilde{f}_i$, $\sigma \in C^2(M, \mathbb{R})$, then ${}^*\sigma \in C^0(\tilde{M}, \mathbb{R})$, such that

$$\sigma(f_i) = {}^*\sigma(\tilde{f}_i).$$

An edge $[v_i, v_j] \in M$ is adjacent to face $f_i, f_j \in M$. Its dual is $[\tilde{f}_i, \tilde{f}_j] \in \tilde{M}$. A 1-form $\omega \in C^1(M, \mathbb{R})$, its Hodge star ${}^*\omega \in C^1(\tilde{M}, \mathbb{R})$,

$${}^*\omega([\tilde{f}_i, \tilde{f}_j]) = \omega([v_i, v_j]).$$

Operators

Exterior differential

$$d : C^k \to C^{k+1}$$

Hodge star operator

$$^* : C^k \to C^{n-k}$$

Coexterior differentiation

$$\delta = {}^*d^* : C^k \to C^{k-1}$$

Laplace operator

$$\Delta := d\delta + \delta d : C^k \to C^k$$

Inner Product

Definition (Inner Product)

Suppose $\omega_1, \omega_2 \in C^1(M, \mathbb{R})$, then define the inner product as

$$\langle \omega_1, \omega_2 \rangle = 2 \sum_{[v_i, v_j] \in M} \omega_1([v_i, v_j]) \omega_2([v_i, v_j]).$$

Lemma

Suppose $\omega_2 \in C^k(M, \mathbb{R})$, $\langle d\omega_1, \omega_2 \rangle = (-1)^k \langle \omega_1, \delta \omega_2 \rangle$.

We call $imgd_0$ as the exact forms, $img\delta_2$ as co-exact forms.

Lemma

Exact forms are orthogonal to co-exact forms.

$$imgd_0 \perp img\delta_2$$

Proof.

Let $f \in C^0$, $g \in C^2$,

$$\langle df, \delta g \rangle = \sum_{[v_i,v_j]\in M} (f(v_j) - f(v_i)(g(f_i) - g(f_j))$$

fix v_i, the faces surrounding v_i are $\{f_0, f_1, \cdots, f_{n-1}\}$, then

$$\sum_{k=0}^{n} f(v_i)(g(f_k) - g(f_{k-1})) = 0$$

Hodge Decomposition

Definition (Harmonic form)

Suppose ω is a k-form, if $\Delta\omega = 0$, then ω is called a harmonic k-form.

Lemma

A k-form ω is harmonic, if and only if $d\omega = 0$ and $\delta\omega = 0$.

Proof.

$\Delta\omega = 0$, then $d\delta\omega = -\delta d\omega$, because exact form is orthogonal to co-exact form, therefore $d\delta\omega = 0$, and $\delta d\omega = 0$. $\langle d\omega, d\omega\rangle = \langle \omega, \delta d\omega\rangle = 0$, therefore $d\omega = 0$. Similarly $\delta\omega = 0$. □

Hodge Decomposition

Suppose τ is a harmonic form, then

$$\langle d\omega, \tau \rangle = \langle \omega, \delta\tau \rangle = 0.$$

$$\langle \delta\omega, \tau \rangle = \langle \omega, d\tau \rangle = 0.$$

Therefore harmonic forms are orthogonal to exact forms and co-exact forms.

Hodge Decomposition

Theorem (Hodge Decomposition)

$$C^k = Imgd_{k-1} \oplus Img\,\delta_{k+1} \oplus \Delta_k$$

where Δ_k is the space of k-harmonic forms.

Proof.

From previous arguments, it is clear that

$$(Imgd_{k-1})^{\perp} \subset Ker\,\delta_k, (Img\,\delta_{k+1})^{\perp} \subset Kerd_k$$

therefore

$$(Imgd_{k-1} \oplus Img\,\delta_{k+1})^{\perp} \subset Ker\,\delta_k \cap Kerd_k = \Delta_k.$$

□