# Exterior Calculus

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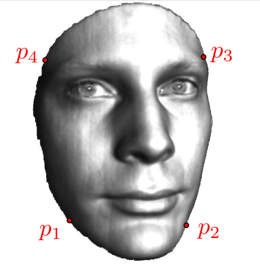
# Conformal Module

## Simply Connected Domains

# Teichmüller Space

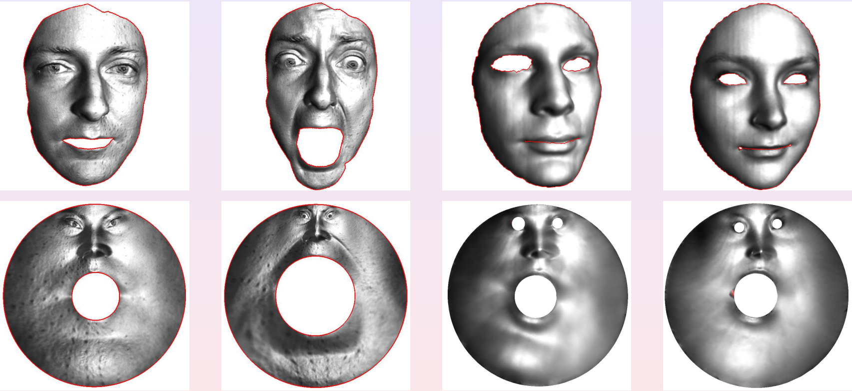
## Topological Quadrilateral




Conformal module: $\frac{h}{w}$. The Teichmüller space is 1 dimensional.

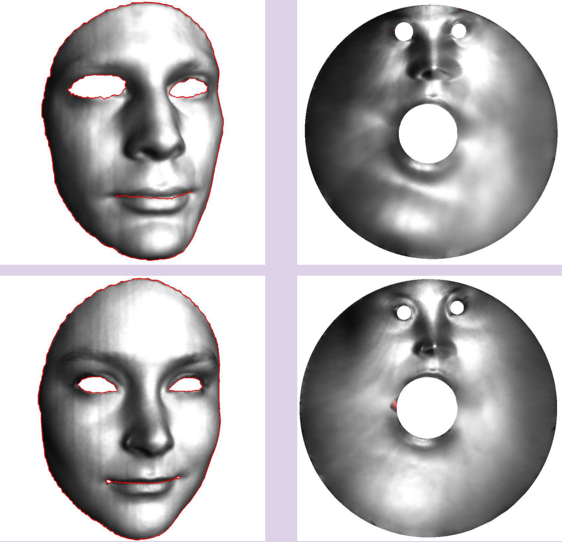
# Conformal Module

## Multiply Connected Domains



# Conformal Module

## Multiply Connected Domains



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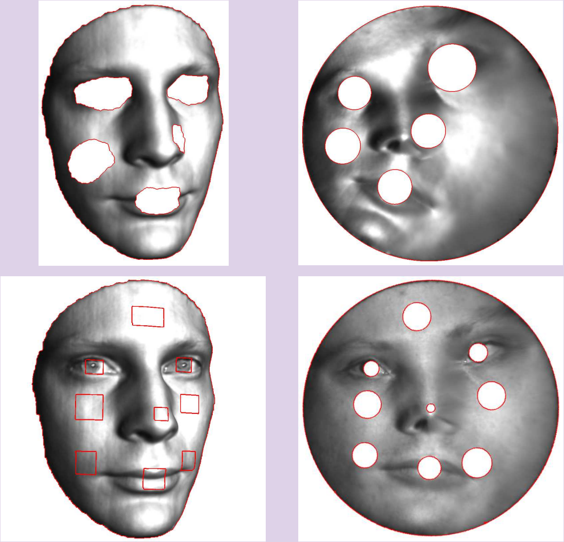
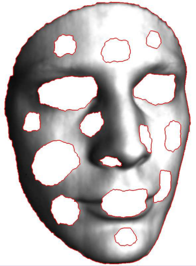
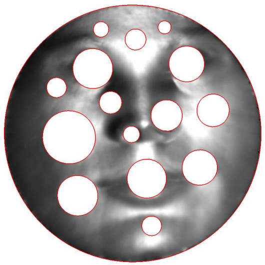
## Multiply Connected Domains



# Teichmüller Space

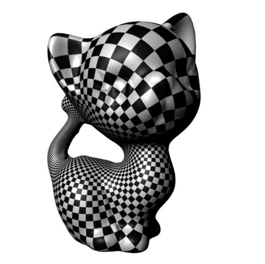
## Multiply Connected Domains

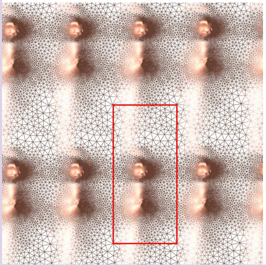




Conformal Module : centers and radii, with Möbius ambiguity. The Teichmüller space is $3n-3$ dimensional, $n$ is the number of holes.

## Torus





# Exterior Calculus

# Holomorphic 1-form

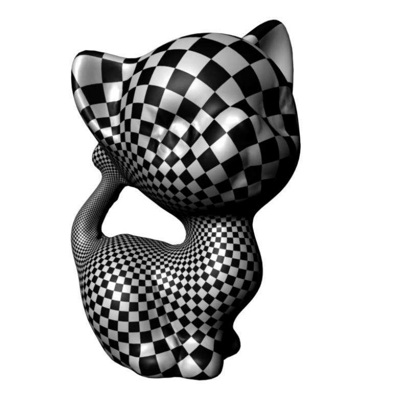
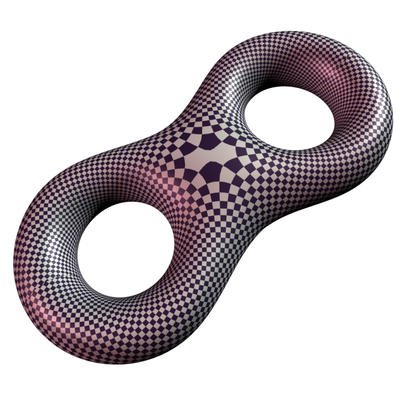


Figure: Holomorphic 1-form

# Holomorphic 1-form



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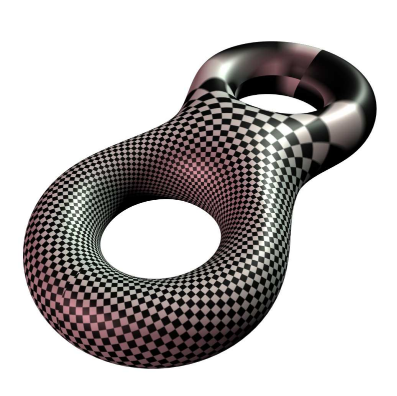

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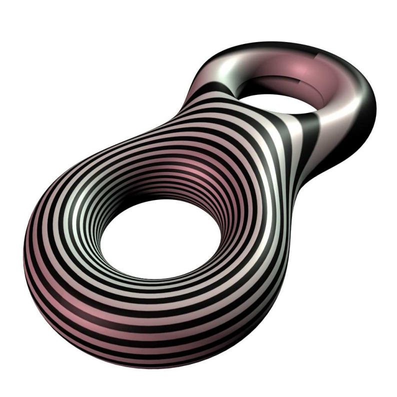

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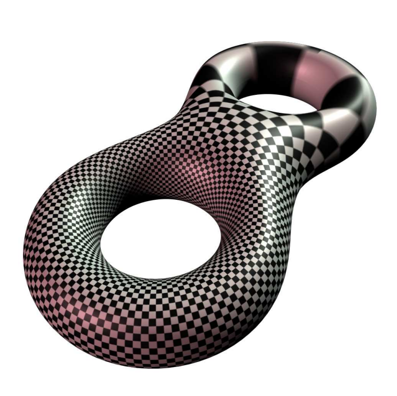

Figure: Holomorphic 1-form

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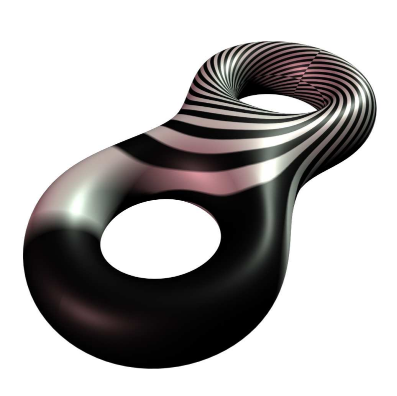

Figure: Holomorphic 1-form

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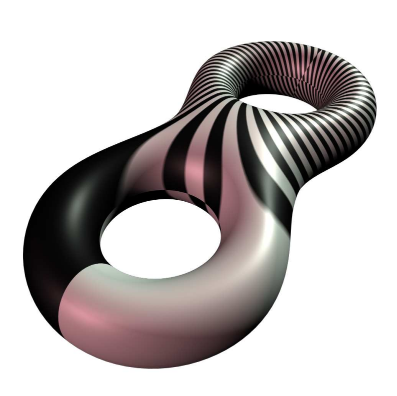

Figure: Holomorphic 1-form

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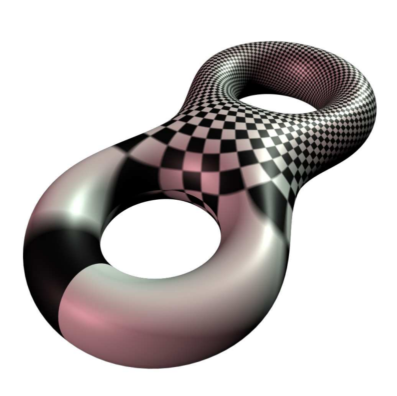

Figure: Holomorphic 1-form

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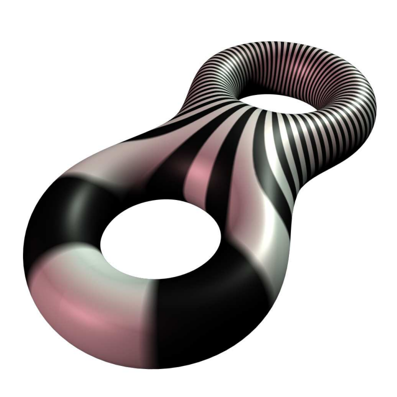

Figure: Holomorphic 1-form

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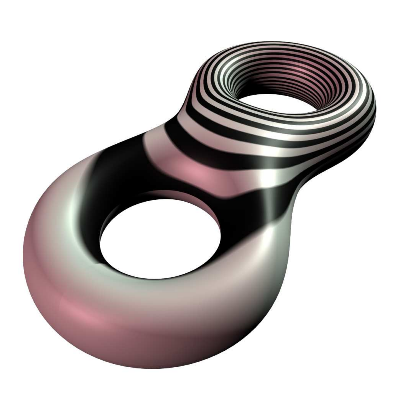

Figure: Holomorphic 1-form

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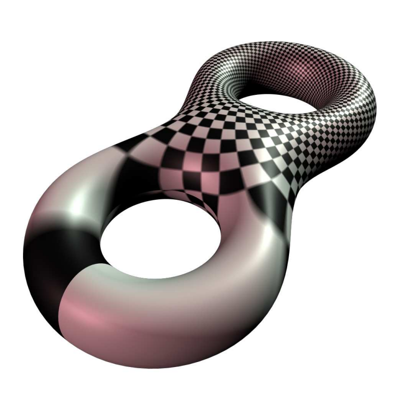

Figure: Holomorphic 1-form

# Philosophy

In each cohomnologous class, there exists a unique harmonic form, which is the smoothest one in the whole class.

Each 1-form is dual to a vector field, the harmonic 1-form corresponds to the vector field, which is with zero curl and zero divergence.

# Smooth manifold

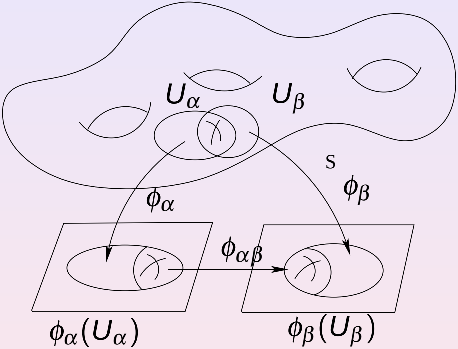


Figure: manifold

**Definition (manifold)**

A manifold is a topological space $M$ covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps $U_\alpha$ to the Euclidean space $\mathbb{R}^n$. $(U_\alpha, \phi_\alpha)$ is called a coordinate chart of $M$. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of $M$. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

If all transition functions $\phi_{\alpha\beta} \in C^\infty(\mathbb{R}^n$ are smooth, then the manifold is a differential manifold or a smooth manifold.

# Tangent Space

## Definition (Tangent Vector)

A tangent vector $\xi$ at the point $p$ is an association to association to every coordinate chart $(x^1, x^2, \cdots, x^n)$ at $p$ an n-tuple $(\xi^1, \xi^2, \cdots, \xi^n)$ of real numbers, such that if $(\tilde{\xi}^1, \tilde{\xi}^2, \cdots, \tilde{\xi}^n)$ is associated with another coordinate system $(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n)$, then it satisfies the transition rule

$$\tilde{\xi}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}(p)\xi^j.$$

A smooth vector field $\xi$ assigns a tangent vector for each point of $M$, it has local representation

$$\xi(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \xi_i(x^1, x^2, \cdots, x^n)\frac{\partial}{\partial x_i}.$$

$\{\frac{\partial}{\partial x_i}\}$ represents the vector fields of the velocities of iso-parametric curves on $M$. They form a basis of all vector

# Push forward

## Definition (Push-forward)

Suppose $\phi : M \to N$ is a differential map from $M$ to $N$, $\gamma : (-\varepsilon, \varepsilon) \to M$ is a curve, $\gamma(0) = p$, $\gamma'(0) = \mathbf{v} \in T_p M$, then $\phi \circ \gamma$ is a curve on $N$, $\phi \circ \gamma(0) = \phi(p)$, we define the tangent vector

$$\phi_*(\mathbf{v}) = (\phi \circ \gamma)'(0) \in T_{\phi(p)} N,$$

as the push-forward tangent vector of $\mathbf{v}$ induced by $\phi$.

**Definition (Differential 1-form)**

The tangent space $T_pM$ is an n-dimensional vector space, its dual space $T_p*M$ is called the cotangent space of $M$ at $p$. Suppose $\omega \in T_p^*M$, then $\omega : T_pM \to \mathbb{R}$ is a linear function defined on $T_pM$, $\omega$ is called a differential 1-form at $p$.

A differential 1-form field has the local representation

$$\omega(x^1,x^2,\cdots,x^n) = \sum_{i=1}^{n} \omega_i(x^1,x^2,\cdots,x^n)dx_i,$$

where $\{dx_i\}$ are the differential forms dual to $\{\frac{\partial}{\partial x_j}\}$, such that

$$dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}.$$

# High order exterior forms

## Definition (Tensor)

A tensor $\Theta$ of type $(m,n)$ on a manifold $M$ is a correspondence that associates to each point $p \in M$ a multi-linear map

$$\Theta_p : T_pM \times T_pM \times \cdots \times T_pM^* \cdots \times T_p^M \to \mathbb{R},$$

where the tangent space $T_pM$ appears $m$ times and cotangent space $T_p^*M$ appears $n$ times.

## Definition (exterior $m$-form)

An exterior $m$-form is a tensor $\omega$ of type $(m,0)$, which is skew symmetric in its arguments, namely

$$\omega_p(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \cdots, \xi_{\sigma(m)}) = (-1)^\sigma \omega_p(\xi_1, \xi_2, \cdots, \xi_m)$$

for any tangent vectors $\xi_1, \xi_2, \cdots, \xi_m \in T_pM$ and any permutation $\sigma \in S_m$, where $S_m$ is the permutation group.

The local representation of $\omega$ in $(x^1, x^2, \cdots, x^m)$ is

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} \omega_{i_1 i_2 \cdots i_m} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m} = \omega_I dx^I,$$

$\omega_I$ is a function of the reference point $p$, $\omega$ is said to be differentiable, if each $\omega_I$ is differentiable.

# Wedge product

**Definition (Wedge product)**

The wedge product of an $m_1$-form and an $m_2$-form $\omega_2$ is an $m_1 + m_2$-form, which is defined in local coordinates by

$$\omega_{I_1} dx^{I_1} \wedge \omega_{I_2} dx^{I_2} = \omega_{I_1} \omega_{I_2} dx^{I_1} dx^{I_2}.$$

A coordinate free representation of wedge product is

$$(\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \cdots, \xi_{m_1+m_2}) = \sum_{\sigma \in S_{m_1+m_2}} \frac{(-1)^\sigma}{m_1! m_2!} \omega_1(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(m_1)}) \omega_2($$

# Pull back

## Definition (Pull back)

Suppose $\phi : M \to N$ is a differentiable map from $M$ to $N$, $\omega$ is an $m$-form on $N$, then the pull-back $\phi^*\omega$ is an $m$-form on $M$ defined by

$$(\phi^*\omega)_p(\xi_1,\cdots,\xi_m) = \omega_{\phi(p)}(\phi_*\xi_1,\cdots,\phi_*\xi_m), p \in M$$

for $\xi_1,\xi_2,\cdots,\xi_m \in T_pM$, where $\phi_*\xi_j \in T_{\phi(p)}N$ is the push forward of $\xi_j \in T_pM$.

Suppose that $U \subset \mathbb{R}^n$ is an open set,

$$\omega = f(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

then

$$\int_U \omega = \int_U f(x)dx^1 dx^2 \cdots dx^n.$$

Suppose $U \subset M$ is an open set of a manifold $M$, a chart $\phi : U \to \Omega \subset \mathbb{R}^n$, then

$$\int_U \omega = \int_\Omega (\phi^{-1})^* \omega.$$

Integration is independent of the choice of the charts. Let $\psi: U \to \psi(U)$ be another chart, with local coordinates $(u_1, u_2, \cdots, u_n)$

$$\int_{\phi(U)} f(x)dx^1dx^2\cdots dx^n = \int_{\psi(U)} f(x(u))det(\frac{\partial x^i}{\partial u^j})du^1du^2\cdots du^n.$$

consider a covering of $M$ by coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ and choose a partition of unity $\{f_i\}$, $i \in I$, such that $f_i(p) \geq 0$,

$$\sum_i f_i(p) \equiv 1, \forall p \in M.$$

Then $\omega_i = f_i \omega$ is an $n$-form on $M$ with compact support in some $U_\alpha$, we can set the integration as

$$\int_M \omega = \sum_i \int_M \omega_i.$$

# Exterior Derivative

Suppose $f : M \to \mathbb{R}$ is a differentiable function, then the exterior derivative of $f$ is a 1-form,

$$df = \sum_i \frac{\partial f}{\partial x_i} dx^i.$$

The exterior derivative of an $m$-form on $M$ is an $(m+1)$-form on $M$ defined in local coordinates by

$$d\omega = d(\omega_I dx^I) = (d\omega_I) \wedge dx^I,$$

where $d\omega_I$ is the differential of the function $\omega_I$.

# Stokes Theorem

**Theorem (Stokes)**

*let $M$ be an n-manifold with boundary $\partial M$ and $\omega$ be a differentialble $(n-1)$-form with compact support on $M$, then*

$$\int_{\partial M} \omega = \int_M d\omega.$$

Let $M$ be a differentiable manifold, $\Omega^n(M)$ represent all the $n$-forms on $M$, $d$ be the exterior derivatives. Then the de Rham complex

$$\cdots \xrightarrow{d^{q-2}} \Omega^{q-1} \xrightarrow{d^{q-1}} \Omega^{q-1} \xrightarrow{d^{q}-1} \Omega^{q} \xrightarrow{d^{q}} \cdots$$

The exterior differentiation operator

$$d^m : \Omega^m(M) \to \Omega^{m+1}(M)$$

is a linear operator with the property

$$d^m \circ d^{m-1} \equiv 0.$$

**Definition (de Rham cohomology group)**

Suppose $M$ is a differential manifold. The $m$-th de Rham cohomology group is defined as

$$H^m_{dR}(M) = \frac{ker d^m}{img d^{m-1}}.$$

**Theorem**

*The de Rham cohomology group $H^m_{dR}(M)$ is isomorphic to the cohomology group $H^m(M,\mathbb{R})$*

$$H^m_{dR}(M) \cong H^m(M,\mathbb{R}).$$

# Hodge Star

Suppose $M$ is a Riemannian manifold, we can locally find oriented orthonormal basis of vector fields, and choose parameterization, such that

$$\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\}$$

form an oriented orthonormal basis. let

$$\{dx_1, dx_2, \cdots, dx_n\}$$

be the dual 1-form basis.

**Definition (Hodge Star Operator)**

The Hodge star opeartor $^* : \Omega^k(M) \to \Omega^{n-k}(M)$ is a linear operator

$$*(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k) = dx_{k+1} \wedge dx_{k+2} \wedge \cdots \wedge dx_n.$$

# Hodge star operator

Let $\sigma = (i_1, i_2, \cdots, i_n)$ be a permutation, then the hoedge star operator

$$*(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = (-1)^{\sigma} dx_{i_{k+1}} \wedge dx_{i_{k+2}} \wedge \cdots \wedge dx_{i_n}.$$

# $L^2$ norm

**Definition**

Let $\eta, \zeta \in \Omega^k(M)$ are two $k$-forms on $M$, then the norm is defined as

$$(\eta, \zeta) = \int_M \eta \wedge {}^*\zeta.$$

$\Omega^k(M)$ is a Hilbert space.

# Codifferential operator

## Definition

The codifferential operator $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined as

$$\delta = (-1)^{k+1+k(n-k)} * d^*,$$

where $d$ is the exterior derivative.

## Lemma

*The codifferential is the adjoint of the exterior derivative, in that*

$$(\delta\zeta, \eta) = (\zeta, d\eta).$$

# Laplace Operator

## Definition (Laplace Operator)

The Laplace operator $\Delta : \Omega^k(M) \to \Omega^k(M)$,

$$\Delta = d\delta + \delta d.$$

## Lemma

*The Laplace operator is symmetric*

$$(\Delta\zeta, \eta) = (\zeta, \Delta\eta)$$

*and non-negative*

$$(\Delta\eta, \eta) \geq 0.$$

## Proof.

$$(\Delta\zeta, \eta) = (d\zeta, d\eta) + (\delta\zeta, \delta\eta).$$

□

# Harmonic Forms

**Definition (Harmonic forms)**

Suppose $\omega \in \Omega^k(M)$, then $\omega$ is called a $k$-harmonic form, if

$$\Delta\omega = 0.$$

**Lemma**

*$\omega$ is a harmonic form, if and only if*

$$d\omega = 0, \delta\omega = 0.$$

**Proof.**

$$0 = (\Delta\omega, \omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega).$$

□

# Hodge Decomposition

## Definition (Harmonic form group)

All harmoic $k$-forms form a group, denoted as $H^k_\Delta(M)$.

## Theorem (Hodge Decomposition)

$$\Omega_k = imgd^{k-1} \bigoplus img\delta^{k+1} \bigoplus H^k_\Delta(M).$$

## Proof.

$(imgd)^\perp = \{\omega \in \Omega^k(M) | (\omega, d\eta) = 0, \forall \eta \in \Omega^{k-1}(M)\}$, because $(\omega, d\eta) = (\delta\omega, \eta)$, so $(imgd^{k-1})^\perp = ker\delta^k$. similarly, $(img\delta^{k+1})^\perp = kerd^k$. Because $imgd^{k-1} \subset kerd^k$, $img\delta^{k+1} \subset ker\delta^k$, therefore $imgd^{k-1} \perp img\delta^{k+1}$,

$$\Omega^k = imgd^{k-1} \oplus img\delta^{k+1} \oplus (imgd^{k-1} \oplus img\delta^{k+1})^\perp$$

$H^k_\Delta = kerd^k \cap ker\delta^k = (imgd^{k-1} \oplus img\delta^{k+1})^\perp$. □

# Hodge Decomposition

suppose $\omega \in kerd^k$, then $\omega \perp img\delta^{k+1}$, then $\omega = \alpha + \beta$, $\alpha \in imgd^{k-1}$, $\beta \in H^k_\Delta(M)$, define project $h : kerd^k \to H^k_\Delta(M)$,

**Theorem**

*Suppose $\omega$ is a closed form, its harmonic component is $h(\omega)$, then the map:*

$$h : H^k_{dR}(M) \to H^k_\Delta(M).$$

*is isomorphic.*

Each cohomologous class has a unique harmonic form.