Holomorphic 1-form method for Conformal Mapping

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Hodge Theory

Harmonic 1-form

Each cohomologous class has a unique harmonic 1-form, which represents a vortex free, source-sink free flow field.

Theorem (Hodge)

All the harmonic 1-forms form a group, which is isomorphic to $H^1(M)$.

Theorem (Hodge Decomposition)

$$\Omega^k(M) = Imgd^{k-1} \oplus Img\delta^{k+1} \oplus H^k_\Delta(M).$$

Compute Harmonic 1-forms

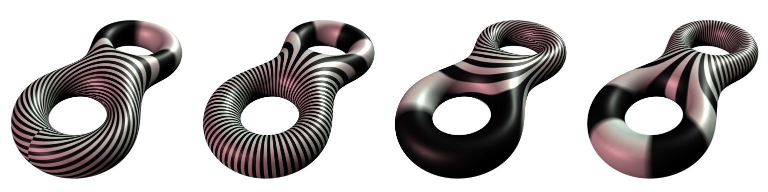
Harmonic 1-form

Let ω be a closed 1-form. Compute a function $f \in C^0(M,\mathbb{R})$, such that

$$\delta^1(\omega + df) = 0,$$

then $\omega + df$ is the unique harmonic 1-form, cohomologous to ω.

Harmonic 1-form Basis



Compute Harmonic 1-forms

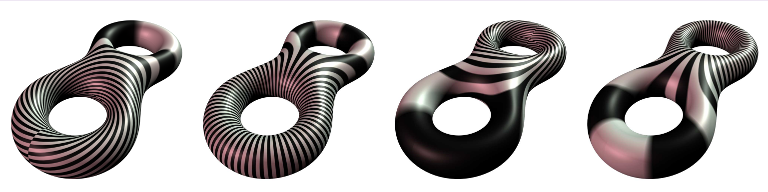
Harmonic 1-form

Let ω be a closed 1-form. Compute a function $f : V \to \mathbb{R}$, such that

$$\sum_j w_{ij}(\omega + df)([v_i, v_j]) = 0, \forall v_i \in V.$$

Then $\omega + df$ is the unique harmonic 1-form, cohomologous to ω.

Harmonic 1-form Basis



Hodge Star Operator

Hodge Star Operator

Let $(S,\mathbf{g})$ be a metric surface, $\{e_1,e_2\}$ be an orthonormal frame field, $\{\frac{\partial}{\partial u},\frac{\partial}{\partial v}\}$ be the base vector fields, $\{du,dv\}$ be the dual differential 1-form fields.

$$^*du = dv, ^*dv = -du.$$

Conjugate harmonic 1-forms $\omega+\sqrt{-1}^*\omega$

Hodge Star Operator

If ω is a harmonic 1-form, so is ${}^*\omega$. Suppose $\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$ is the set basis of harmonic 1-forms, then ${}^*\omega = \sum_k \lambda_k \omega_k$. Locally, on each triangle ${}^*(adx + bdy) = ady - bdx$. Solve linear system

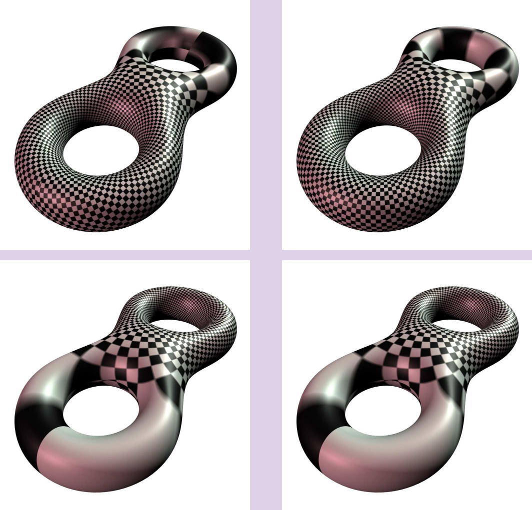
$$\int_M \omega_i \wedge {}^*\omega = \sum_k \lambda_k \int_M \omega_i \wedge \omega_k, i = 1, 2, \cdots, 2g.$$

to solve λ_k's, where ${}^*\omega$ on the left hand side is locally evaluated.

Conjugate harmonic 1-forms $\omega + \sqrt{-1}{}^*\omega$

Holomorphic 1-form

Holomorphic 1-form Basis



Topological Quadrilateral

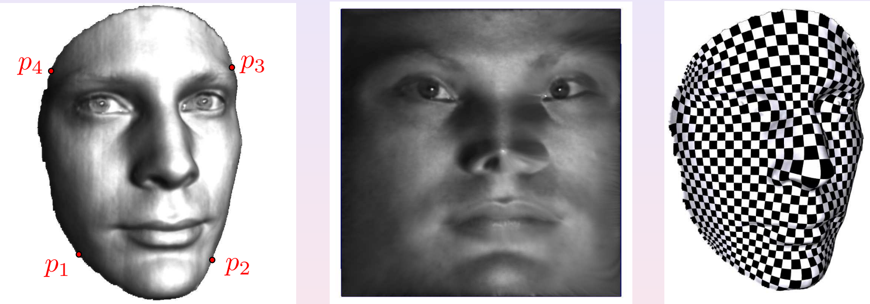


Figure: Topological quadrilateral.

Topological Quadrilateral

Definition (Topological Quadrilateral)

Suppose S is a surface of genus zero with a single boundary, and four marked boundary points $\{p_1, p_2, p_3, p_4\}$ sorted counter-clock-wisely. Then S is called a topological quadrilateral, and denoted as $Q(p1, p2, p3, p4)$.

Theorem

Suppose $Q(p1, p2, p3, p4)$ is a topological quadrilateral with a Riemannian metric $\mathbf{g}$, then there exists a unique conformal map $\phi : S \to \mathbb{C}$, such that ϕ maps Q to a rectangle, $\phi(p_1) = 0$, $\phi(p_2) = 1$. The height of the image rectangle is the conformal module of the surface.

Topological Quadrilateral

Assume the boundary of Q consists of four segments $\partial Q = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, such that

$$\partial\gamma_1 = p_2 - p_1, \partial\gamma_2 = p_3 - p_2, \partial\gamma_3 = p_4 - p_3, \gamma_4 = p_1 - p_4.$$

We compute two harmonic functions $f_1, f_2 \to \mathbb{R}$, such that

$$\begin{cases} \Delta f_1 & = & 0 \\ f_1|_{\gamma_1} & = & 0 \\ f_1|_{\gamma_3} & = & 1 \\ \frac{\partial f_1}{\partial \mathbf{n}}|_{\gamma_2 \cup \gamma_4} & = & 0 \end{cases} \begin{cases} \Delta f_2 & = & 0 \\ f_2|_{\gamma_2} & = & 0 \\ f_2|_{\gamma_4} & = & 1 \\ \frac{\partial f_2}{\partial \mathbf{n}}|_{\gamma_1 \cup \gamma_3} & = & 0 \end{cases}$$

Topological Quadrilateral

The df_1 and df_2 are two exact harmonic 1-forms. We need to find a scalar λ, such that ${}^*df_1 = \lambda df_2$, this can be achieved by solving the following equation,

$$\int_S df_1 \wedge {}^*df_2 = \lambda \int_S df_1 \wedge df_2.$$

Then the desired holomorphic 1-form $\omega = df_1 + i\lambda df_2$. The conformal mapping is given by

$$\phi(p) = \int_q^p \omega,$$

where q is the base point, the path from q to p is arbitrarily chosen.

Topological Annulus

Topological Annulus

Figure: Topological annulus.

Topological Annulus

Definition (Topological Annulus)

Suppose S is a surface of genus zero with two boundaries, the S is called a topological annulus.

Theorem

Suppose S is a topological annulus with a Riemannian metric $\mathbf{g}$, the boundary of S are two loops $\partial S = \gamma_1 - \gamma_2$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical annulus, $\phi(\gamma_1)$ is the unit circle, $\phi(\gamma_2)$ is another concentric circle with radius γ. Then $-\log\gamma$ is the conformal module of S. The mapping ϕ is unique up to a planar rotation.

Topological Annulus

First, we compute a harmonic function $f : S \to \mathbb{R}$, such that

$$\begin{cases} f|_{\gamma_1} & = & 0 \\ f|_{\gamma_2} & = & 1 \\ \Delta f & = & 0 \end{cases}$$

Then df is an exact harmonic 1-form. Then we compute a harmonic 1-form τ, such that $\int_{\gamma_1} \tau = 1$.

Topological Annulus

Then we compute a constant λ, such that ${}^{*}df = \lambda\tau$, by solving the following equation,

$$\int_S df \wedge {}^{*}df = \lambda \int_S df \wedge \tau.$$

Then $\omega = df + i\lambda\tau$ is a holomorphic 1-form. Let $Img(\int_{\gamma_1} \omega) = k$. The conformal mapping is given by

$$\phi(p) = exp^{\frac{2\pi}{k}\int_q^p \omega}.$$

Topological Annulus

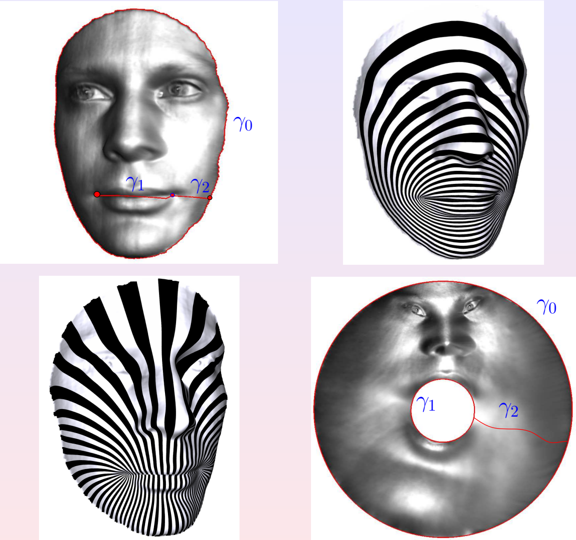


Figure: Topological annulus.

Riemann Mapping

Conformal Module

Simply Connected Domains

Topological Disk

Definition (Topological Disk)

Suppose S is a surface of genus zero with one boundary, the S is called a topological disk.

Theorem

Suppose S is a topological disk with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical disk. The mapping ϕ is unique up to a Möbius transformation,

$$z \to e^{i\theta}\frac{z - z_0}{1 - \bar{z}_0 z}.$$

Topological Disk

Punch a small hole in the disk, then use the algorithm for topological annulus to compute the conformal mapping. The punched hole will be mapped to the center.

Multiply connected domains

Multiply-Connected Annulus

Definition (Multiply-Connected Annulus)

Suppose S is a surface of genus zero with multiple boundaries, then S is called a multiply connected annulus.

Theorem

Suppose S is a multiply connected annulus with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the unit disk with circular holes. The radii and the centers of the inner circles are the conformal module of S. Such kind of conformal mapping are unique up to Möbius transformations.

Conformal Slit Mapping

Figure: Harmonic forms and holomorphic forms.

Conformal Slit Mapping

Slit Mapping

Suppose there are $n+1$ boundary components$\{\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_n\}$. $\{\omega_1, \omega_2, \cdots, \omega_n\}$ are the holomorphic 1-form basis. Choose two boundary components, γ_0, γ_1, solve linear equation $\omega = \sum_{k=1}^{n} \lambda_k \omega_k$,

$$img(\int_{\gamma_0} \omega) = 2\pi, img(\int_{\gamma_1} \omega) = -2\pi, img(\int_{\gamma_k} \omega) = 0, 2 \leq k \leq n.$$

Then the mapping is given by

$$p \to \exp \int_q^p \omega,$$

where q is the base point on the surface.

Conformal Circular Slit Mapping

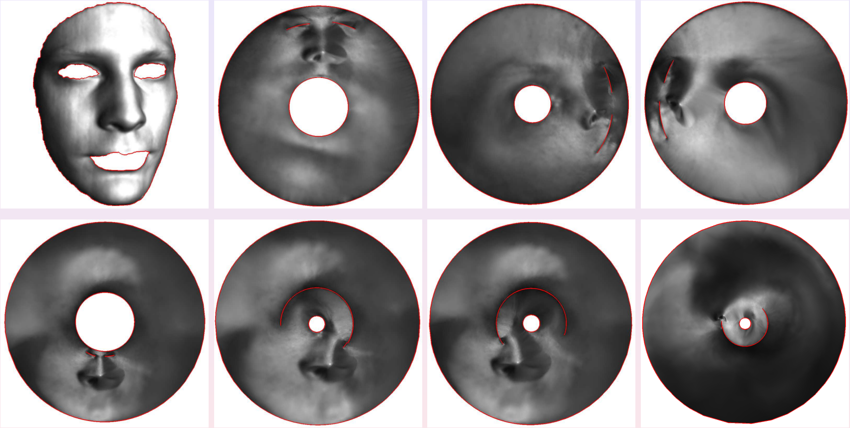


Figure: Conformal circular slit mapping.

Hole Filling

Adding sample points in the center hole, use Delaunay triangulation to fill in with boundary constraints.

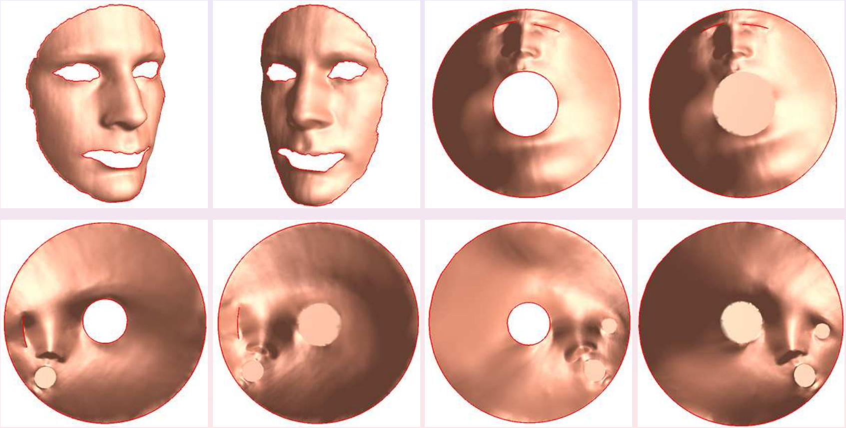


Figure: Fill interior holes.

Koebe's Iteration - I

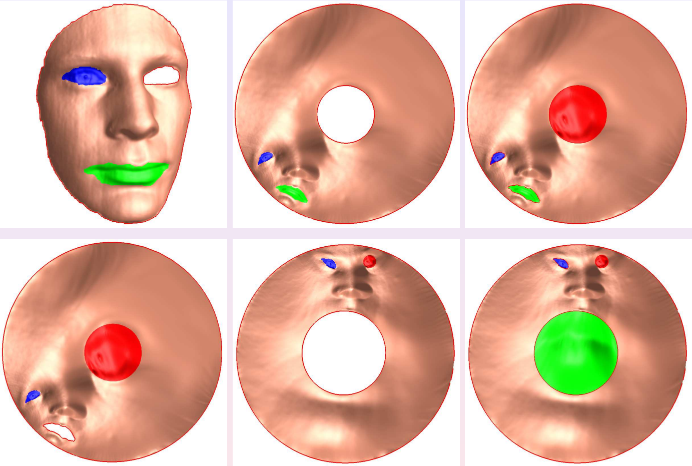


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Koebe's Iteration - II

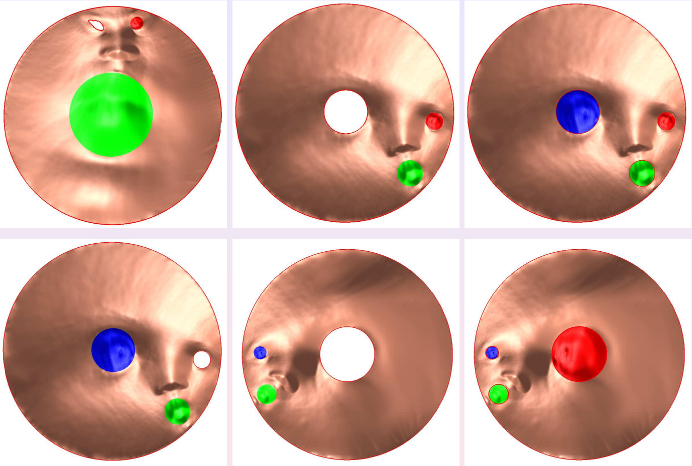


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Koebe's Iteration - III

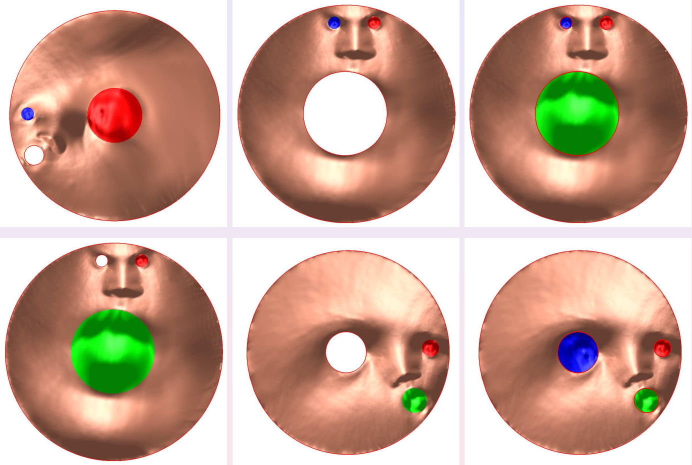


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Convergence Analysis

Theorem (Gu and Luo 2009)

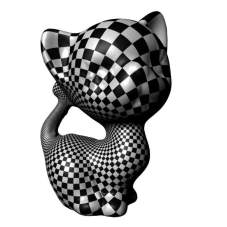
Suppose genus zero surface has n boundaries, then there exists constants $C_1 > 0$ and $0 < C_2 < 1$, for step k, for all $z \in \mathbb{C}$,

$$|f_k \circ f^{-1}(z) - z| < C_1 C_2^{2[\frac{k}{n}]},$$

where f is the desired conformal mapping.

Topological Torus

Topological torus



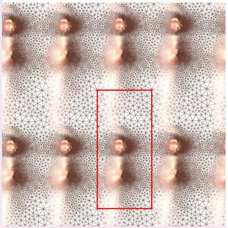


Figure: Genus one closed surface.

Topological Torus

1. We compute a basis for the fundamental group $\pi_1(S)$, $\{\gamma_1, \gamma_2\}$.
2. Compute the holomorophic 1-form basis ω_1, ω_2, such that $\int_{\gamma_i} \omega_j = \delta_{ij}$.
3. Slice the surface along γ_1, γ_2 to get a fundamental domain $\tilde{S}$,
4. The conformal mapping $\phi : \tilde{S} \to \mathbb{C}$ is given by

$$\phi(p) = \int_q^p \omega_1,$$

where q is the base point, the path from q to p in $\tilde{S}$ can be arbitrarily chosen.

Topological Torus

Suppose $a+ib=\int_{\gamma_2}\omega_1$, then $a+ib$ is the conformal module of the torus. The deck transformation group generators are

$$T_1(z)=z+1, T_2(z)=z+a+ib.$$

By using all deck transformations to translate $\phi(\tilde{S})$, we can conformally map the universal covering space of S onto the whole complex plane $\mathbb{C}$, each fundamental domain is a parallelogram.