Spherical Conformal Mapping

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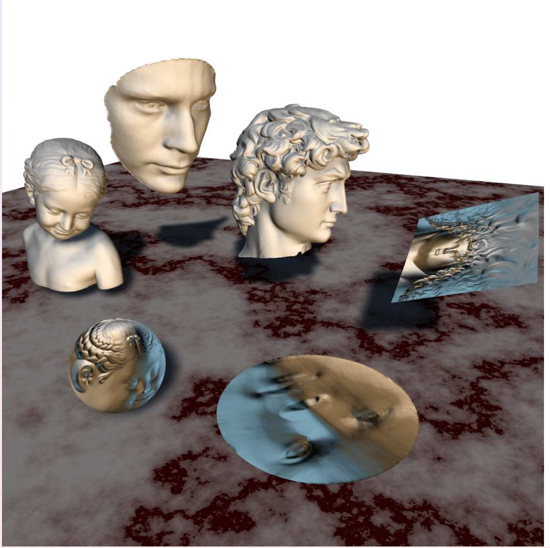
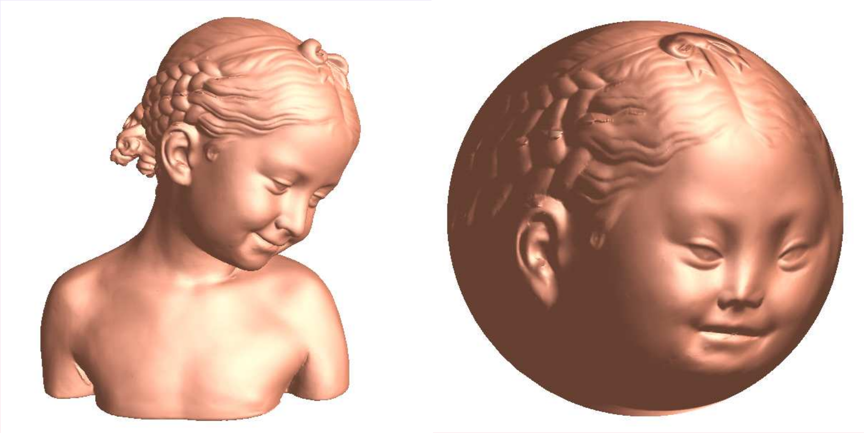
Harmonic Map

Surface Parameterization

Map the surfaces onto canonical parameter domains



Spherical harmonic map



Let (M,g) and (N,h) be Riemannian manifolds, $u: M \to N$ is a C^1 mapping.

$$ds_M^2 = \sum g_{\alpha\beta} dx^\alpha dx^\beta, ds_N^2 = \sum h_{ij}(u(x)) du^i du^j.$$

The pull back metric of h induced by u is $u^*(ds_N^2)$ is a symmetric bilinear form

$$u^*(dS_N^2) = \sum_{\alpha,\beta} (\sum_{i,j} h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}) dx^\alpha dx^\beta.$$

The *energy density* of mapping u is defined as

$$|du|^2 = \sum_{i,j,\alpha,\beta} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

Energy of the mapping

Equivalently, choose an orthogonal frame field under $u^*(ds_N^2)$, each basis vector field is of unit length under **g**, the dual 1-forms are $\{\omega_1, \omega_2, \cdots, \omega_n\}$, such that

$$u^*(ds_N^2) = \sum_{\alpha=1}^{n} \lambda_\alpha (\omega_\alpha)^2.$$

The the energy density of the mapping u is given by

$$|du|^2 = \sum_{\alpha=1}^{n} \lambda_\alpha.$$

Harmonic Energy and Harmonic Mapping

Definition (Harmonic Energy)

The harmonic energy functional $E(u)$ is defined as

$$E(u) = \int_M |du|^2 dv_M,$$

where $dv_M = (detg)^{\frac{1}{2}} dx$ is the volume element of M.

Definition (Harmonic Mapping)

In the space of mappings, the critical points of $E(u)$ are called harmonic mappings.

Harmonic Energy Conformal Invariant

Suppose u is a mapping from a surface (S,g) to (N,h). Suppose $\tilde{g} = e^{2\lambda}g$ is another metric of S, conformal to g, then

$$|\tilde{d}u|^2 = e^{-2\lambda}|du|^2, \sqrt{det\tilde{g}} = e^{2\lambda}\sqrt{detg},$$

Then $\tilde{g} = g$. Harmonic energy is invariant under conformal metric transformation.

Theorem

Harmonic energy only depends on the conformal structure of the surface, independent of the choice of Riemannian metric.

Harmnonic function

Suppose $\Omega \subset \mathbb{R}^2$ is a planar domain, $f : \Omega \to \mathbb{R}$ is a function. The gradient of f is given by

$$\nabla f := (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})^T.$$

The harmonic energy of the function is given by

$$E(f) := \int_\Omega \langle \nabla f, \nabla f \rangle dxdy.$$

Harmonic function satisfies

$$\Delta f = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) f = 0,$$

where Δ is the Laplace-Beltrami operator.

$$\Delta = \nabla \cdot \nabla,$$

the divergence of gradient.

Harmnonic function

Suppose (S, g) is a surface. We choose the isothermal coordinates

$$g = e^{2\lambda(x,y)}(dx^2 + dy^2).$$

Suppose $f : S \to \mathbb{R}$ is a function defined on S. The gradient of f is defined as

$$\nabla_g f := e^{-\lambda} \nabla f.$$

The harmonic energy density is

$$|df|^2 = e^{-2\lambda(x,y)}[(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2].$$

The area element

$$e^{2\lambda} dxdy.$$

The Laplace Beltrami operator is given by

$$\Delta_g := e^{-2\lambda} \Delta$$

Theorem

suppose $f : S \to \mathbb{R}$ is a harmonic function, f is zero on the boundary of S, $f|_{\partial S} = 0$, then $\Delta_g f = 0$.

Proof.

$$\frac{d}{d\varepsilon}\int_S <\nabla(f+\varepsilon g), \nabla(f+\varepsilon g)> dv_S = \int_S <\nabla f, \nabla g> dv_S = 0.$$

choose an arbitrary function $g : S \to \mathbb{R}$, such that the restriction of g on ∂S is 0. $\nabla \cdot (g\nabla f) =< \nabla g, \nabla f > + g\Delta f$.

$$\int_S <\nabla f, \nabla g> dv_S = \int_S \nabla \cdot (g\nabla f) - g\Delta f = \int_{\partial S} g\nabla f - \int_S g\Delta f$$

□

Suppose N is embedded in $\mathbb{R}^3$, $u : S \to N$ is a harmonic mapping, then

$$\Delta_g u^{T_u N} \equiv 0.$$

where $\Delta_g u = (\Delta_g u_1, \Delta_g u_2, \Delta_g u_3)$. Namely, $\Delta_g u$ is orthogonal to the tangent plane at the target space.

Heat Flow method

Definition (Heat flow)

Let $u : S \rightarrow N \subset \mathbb{R}^3$, the heat flow is given by

$$\frac{du(x,t)}{dt} = -(\Delta_g u)^{T_{u(x)}N}$$

The heat flow method will deform a mapping to the harmonic mapping under special normalization conditions.

Heat Flow method

Theorem

Harmonic mapping from a genus zero closed surface to the unit sphere must be a conformal mapping.

Proof.

Let $u : S \to \mathbb{S}^2$. Choose isothermal coordinates of both surfaces, define

$$\phi(z) = \langle \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} \rangle$$

then

$$\phi(z) = \frac{1}{4}(|\frac{\partial u}{\partial x}|^2 - |\frac{\partial u}{\partial y}|^2 - \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle).$$

if $\phi(z) = 0$, then the mapping is conformal.
On the other hand, $\frac{\partial \phi(z)}{\partial \bar{z}} = 0$, then $\phi(z)$ is holomorphic.
$\phi(z)dz^2$ is globally defined, the so-called Hopf differential.
Sphere has no non-zero holomorphic quadratic differentials.

Möbius transformation

Theorem

The conformal automorphism from a sphere to itself must be a Möbius transformation

$$z \to \frac{az+b}{cz+d}, ad-bc=1, a,b,c,d \in \mathbb{C}.$$

Rado's theorem

Theorem (Rado)

Let $\Omega \subset \mathbb{R}^2$ is a convex domain with smooth boundary. For any homeomorphism $\phi : S^1 \to \partial\Omega$, there exists a unique harmonic mapping $u : D \to \Omega$, such that $u|_{\partial}D = \phi$, furthermore, u is a diffeomorphism.

We use piecewise linear triangle mesh to approximate the original surface. suppose $u : M \to \mathbb{R}$ the harmonic energy is given by

$$E(u) = \frac{1}{2} \sum_{[v_i,v_j]\in M} w_{ij}(f(v_i) - f(v_j))^2.$$

The discrete Laplace-Beltrami operator is given by

$$\Delta f(v_i) = \sum_j w_{ij}(f(v_j) - f(v_i)).$$

where w_{ij} is the cotangent formula.