Yambe Equation

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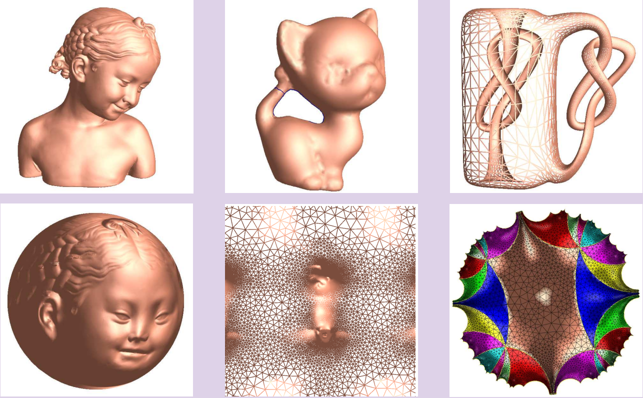
Orthonormal Movable Frame

Canonical Conformal Representations

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical Euclidean Hyperbolic

Orthonormal Movable frame

Suppose a regular surface S is embedded in $\mathbb{R}^3$, a parametric representation is $\mathbf{r}(u,v)$. Select two vector fields $\mathbf{e}_1, \mathbf{e}_2$, such that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}.$$

Let $\mathbf{e}_3$ be the unit normal field of the surface. Then

$$\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

form the *orhonormal frame field* of the surface.

Orthonormal Movalbe frame

The tangent vector is the linear combination of the frame bases,

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2$$

where $\omega_k(\mathbf{v}) = \langle \mathbf{e}_k, \mathbf{v} \rangle$. $d\mathbf{r}$ is orthogonal to the normal vector $\mathbf{e}_3$.

$$d\mathbf{e}_i = \omega_{i1}\mathbf{e}_1 + \omega_{i2}\mathbf{e}_2 + \omega_{i3}\mathbf{e}_3,$$

where $\omega_{ij} = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle$. Because

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij},$$

we get

$$\omega_{ij} + \omega_{ji} = 0, \omega_{ii} = 0.$$

Motion Equation

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2,$$

$$\begin{pmatrix} d\mathbf{e}_1 \\ d\mathbf{e}_2 \\ d\mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

The first fundamental form is

$$I = \langle d\mathbf{r}, d\mathbf{r} \rangle = \omega_1 \omega_1 + \omega_2 \omega_2.$$

The second fundamental form is

$$II = -\langle d\mathbf{r}, d\mathbf{e}_3 \rangle = -\omega_1 \omega_{31} - \omega_2 \omega_{32} = \omega_1 \omega_{13} + \omega_2 \omega_{23}.$$

Weingarten Mapping

Definition (Weingarten Mapping)

The Gauss mapping is

$$\mathbf{r} \to \mathbf{e}_3,$$

its derivative map is called the weingarten mapping,

$$d\mathbf{r} \to d\mathbf{e}_3, \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 \to \omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2.$$

Definition (Gaussian Curvature)

The area ratio (Jacobian of the Weingarten mapping) is the Gaussian curvature

$$K\,\omega_1 \wedge \omega_2 = \omega_{31} \wedge \omega_{32}.$$

Gaussian curvature

$\{\omega_1, \omega_2\}$ form the basis of the cotangent space, therefore ω_{13}, ω_{23} can be represented as the linear combination of them,

$$\begin{pmatrix} \omega_{13} \\ \omega_{31} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

therefore

$$\omega_{13} \wedge \omega_{23} = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \omega_1 \wedge \omega_2$$

so $K = h_{11}h_{22} - h_{12}h_{21}$, the mean curvature $H = \frac{1}{2}(h_{11} + h_{22})$.

Gauss's theorem Egregium

$$
\begin{aligned}
0 &= d^2\mathbf{e}_1 \\
&= d(\omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3) \\
&= d\omega_{12}\mathbf{e}_2 - \omega_{12}\wedge d\mathbf{e}_2 + d\omega_{13}\mathbf{e}_3 - \omega_{13}\wedge d\mathbf{e}_3 \\
&= d\omega_{12}\mathbf{e}_2 - \omega_{12}\wedge(\omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3) + \\
&\quad d\omega_{13}\mathbf{e}_3 - \omega_{13}\wedge(\omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2) \\
&= (d\omega_{12} - \omega_{13}\wedge\omega_{32})\mathbf{e}_2 + (d\omega_{13} - \omega_{12}\wedge\omega_{23})\mathbf{e}_3 +
\end{aligned}
$$

therefore

$$d\omega_{12} = -\omega_{13}\wedge\omega_{23} = -K\omega_1\wedge\omega_2.$$

Theorem (Gauss' Theorem Egregium)

The Gaussian curvature is intrinsic, solely determined by the first fundamental form.

$$
\begin{aligned}
0 &= d^2\mathbf{r} \\
&= d(\omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2) \\
&= d\omega_1\mathbf{e}_1 - \omega_1 \wedge d\mathbf{e}_1 + d\omega_2\mathbf{e}_2 - \omega_w \wedge d\mathbf{e}_2 \\
&= d\omega_1\mathbf{e}_1 - \omega_1 \wedge (\omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3) + \\
& \quad d\omega_2\mathbf{e}_2 - \omega_2 \wedge (\omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3) \\
&= (d\omega_1 - \omega_2 \wedge \omega_{21})\mathbf{e}_1 + (d\omega_2 - \omega_1 \wedge \omega_{12})\mathbf{e}_2 + \\
& \quad -(\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23})\mathbf{e}_3.
\end{aligned}
$$

therefore

$$
\begin{cases}
d\omega_1 = \omega_2 \wedge \omega_{21}, \\
d\omega_2 = \omega_1 \wedge \omega_{12}
\end{cases}
$$

Gaussian Curvature

Let $(S,\mathbf{g})$ be a metric surface, use isothermal coordinates

$$\mathbf{g} = e^{2u(x,y)}(dx^2+dy^2).$$

Then

$$\begin{cases} \omega_1 & = & e^u dx \\ \omega_2 & = & e^u dy \end{cases} \quad \begin{cases} \mathbf{e}_1 & = & e^{-u}\frac{\partial}{\partial x} \\ \mathbf{e}_2 & = & e^{-u}\frac{\partial}{\partial y} \end{cases}$$

$$\begin{aligned} d\omega_1 &= de^u \wedge dx \\ &= e^u(u_x dx + u_y dy) \wedge dx \\ &= e^u u_y dy \wedge dx. \end{aligned}$$

$$\begin{aligned} d\omega_2 &= de^u \wedge dy \\ &= e^u(u_x dx + u_y dy) \wedge dy \\ &= e^u u_x dx \wedge dy. \end{aligned}$$

Assume $\omega_{12} = a\omega_1 + b\omega_2$, then

$$\begin{aligned} d\omega_1 &= -\omega_2 \wedge \omega_{12} = a\omega_1 \wedge \omega_2 \\ e^u u_y dy \wedge dx &= ae^{2u} dx \wedge dy \end{aligned}$$

$a = -e^{-u} u_y$.

Gaussian Curvature

$$\begin{aligned} d\omega_2 &= \omega_1 \wedge \omega_{12} = b\omega_1 \wedge \omega_2 \\ e^u u_x dx \wedge dy &= be^{2u} dx \wedge dy \end{aligned}$$

$b = e^{-u} u_x$. therefore

$$\omega_{12} = -u_y dx + u_x dy$$

$$\begin{aligned} d\omega_{12} &= (u_{xx} + u_{yy}) dx \wedge dy \\ &= -K\omega_1 \wedge \omega_2 \\ &= -Ke^{2u} dx \wedge dy. \end{aligned}$$

Gaussian curvature

$$K = -\frac{1}{e^{2u}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u.$$

Gaussian Curvature

Example

The unit disk $|z| < 1$ equipped with the following metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2},$$

the Gaussian curvature is -1 everywhere.

Proof.

$e^{2u} = \frac{4}{1-x^2-y^2}$, then $u = \log 2 - \log(1 - x^2 - y^2)$.

$$u_x = -\frac{-2x}{1 - x^2 - y^2} = \frac{2x}{1 - x^2 - y^2}.$$

□

Proof.

then

$$u_{xx} = \frac{2(1-x^2-y^2)-2x(-2x)}{(1-x^2-y^2)^2} = \frac{2+2x^2-2y^2}{(1-x^2-y^2)^2}$$

similarly

$$u_{yy} = \frac{2+2y^2-2x^2}{(1-x^2-y^2)^2}$$

so

$$u_{xx}+u_{yy} = \frac{4}{(1-x^2-y^2)} = e^{2u}, K = -\frac{1}{e^{2u}}(u_{xx}+u_{yy}) = -1.$$

□

Yamabe Equation

Conformal metric deformation $\mathbf{g} \to e^{2\lambda}\mathbf{g} = \tilde{\mathbf{g}}$

$$\mathbf{g} = e^{2u}(dx^2 + dy^2), K = -e^{2u}\Delta u.$$

similarly

$$\tilde{\mathbf{g}} = e^{2\tilde{u}}(dx^2 + dy^2), \tilde{K} = -e^{2\tilde{u}}\Delta \tilde{u}.$$

$$\begin{aligned}
\tilde{K} &= -\frac{1}{e^{2(u+\lambda)}}\Delta(u+\lambda) \\
&= \frac{1}{e^{2\lambda}}\left(-\frac{1}{e^{2u}}\Delta u - \frac{1}{e^{2u}}\Delta\lambda\right) \\
&= \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).
\end{aligned}$$

Yamabe Equation

Yamabe Equation

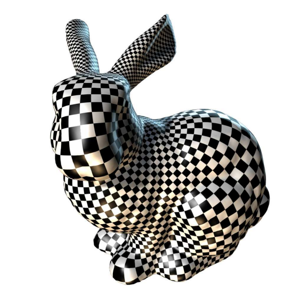
$$\tilde{K} = \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda), \tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}.$$

Discrete Curvature Flow

Isothermal Coordinates

A surface Σ with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Gaussian Curvature

The Gaussian curvature is given by

$$K(u,v) = -\Delta_{\mathbf{g}}\lambda = -\frac{1}{e^{2\lambda(u,v)}}\Delta\lambda(u,v),$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Conformal Metric Deformation

Definition

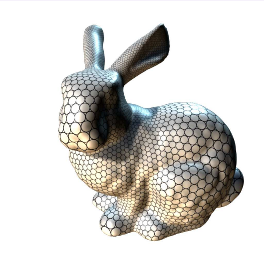
Suppose Σ is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. λ is called the conformal factor.

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Yamabe Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(-\Delta_{\mathbf{g}}\lambda + K),$$

geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(-\partial_n\lambda + k_g).$$

Surface Ricci Flow

Definition (Hamilton's Surface Ricci Flow)

A closed surface with a Riemannian metric **g**, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -Kg_{ij}.$$

If the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant every where.

Key Idea

$K = -\Delta_{\mathbf{g}}\lambda$, Roughly speaking, $\frac{dK}{dt} = \Delta_{\mathbf{g}}\frac{d\lambda}{dt}$. Let $\frac{d\lambda}{dt} = -K$, then

$$\frac{dK}{dt} = \Delta_{\mathbf{g}}K$$

Heat equation!

Theorem (Hamilton 1988)

For a closed surface of non-positive Euler characteristic number, the normalized Ricci flow

$$\frac{d\lambda(t)}{dt} = -2K(t) - \frac{2\pi\chi(S)}{A(t)}$$

will converges to a metric such that the Gaussian curvature is constant everywhere.

Theorem (Chow 1991)

For a closed surface of positive Euler characteristic number, the normalized Ricci flow will converges to a metric such that the Gaussian curvature is constant everywhere.