Heat Diffusion

David Gu[1,2]

[1]Computer Science Department
Stony Brook University
[2]Yau Mathematical Sciences Center
Tsinghua University

Tsinghua University

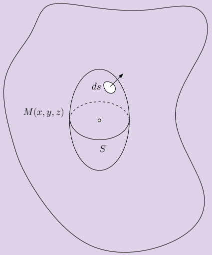
Heat Diffusion

Heat Flow

Physics

Assume an object in $\mathbb{R}^3$ has temperature $u(x,y,z,t)$ at the point $M(x,y,z)$, $k(x,y,z)$ represents the heat conductivity coefficient, $c(x,y,z)$ represents the heat capacity ratio, $\rho(x,y,z)$ represents the material density.

Configuration



Fourier's law of conduction

Theorem (Fourier's law of conduction)

within a infinitesimal small time period dt, the heat transfer dQ through an infinitesimal area ds is proportional to the ***negative*** *directional derivative of the temperature of the object* $\frac{\partial u}{\partial \mathbf{n}} = \langle \nabla u, \mathbf{n} \rangle$,

$$dQ = -k(x,y,z)\frac{\partial u}{\partial \mathbf{n}} ds dt,$$

where $\mathbf{n}$ *is the normal to the area element* ds*, the gradient* $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z})$.

The heat transfer always flows from the high temperature area to the low temperature area; the gradient always from low temperature area points to the hight temperature area.

Fourier's law of conduction

Get a closed surface S inside the object, the volume inside is denoted as Ω, then from time t_1 to time t_2 the heat transfer through the surface S entering Ω is

$$Q_1 = \int_{t_1}^{t_2} \int_S k \frac{\partial u}{\partial \mathbf{n}} dsdt,$$

According to Stokes theorem

$$Q_1 = \int_{t_1}^{t_2} \int_\Omega k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y} + \frac{\partial^2 u}{\partial z^2}\right) dvdt.$$

Heat Transfer Equation

At every point inside Ω, from t_1 to t_2 the temperature change needs the heat

$$Q = \int_{t_1}^{t_2} \int_{\Omega} c(x,y,z)\rho(x,y,z)\frac{\partial u}{\partial t} dvdt.$$

Because Ω and the time interval $[t_1, t_2]$ are arbitrary, let $a^2 = \frac{k}{c\rho}$, then we get the heat transfer equation

Definition (Heat Diffusion Equation)

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$