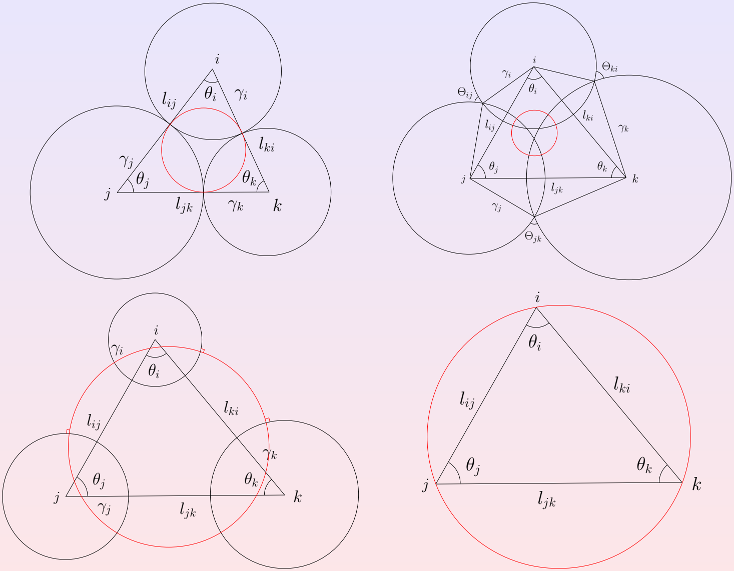
Inversive Distance Euclidean Curvature Flows

David Gu[1,2]

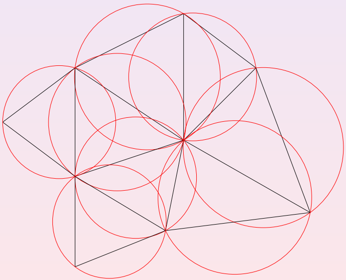
[1]Computer Science Department
Stony Brook University
[2]Yau Mathematical Sciences Center
Tsinghua University

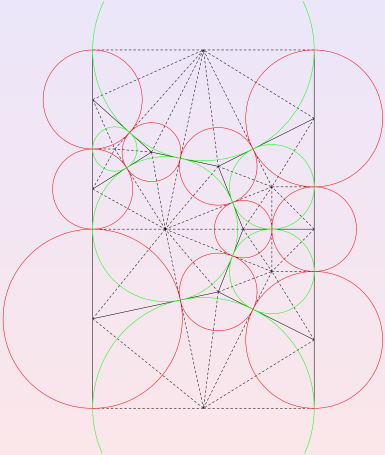
Tsinghua University

Circle Patterns

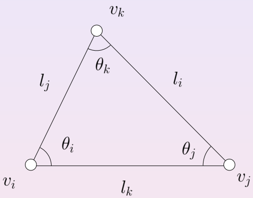


Circle Patterns





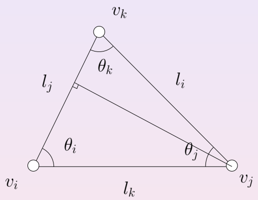
Cosine law



$$A = l_j l_k \sin\theta_i$$

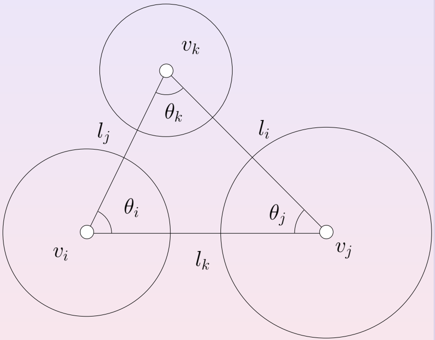
$$\begin{aligned} 2 l_j l_k \cos\theta_i &= l_j^2 + l_k^2 - l_i^2 \\ -2 l_j l_k \sin\theta_i \frac{d\theta_i}{dl_i} &= -2 l_i \\ \frac{d\theta_i}{dl_i} &= \frac{l_i}{A} \end{aligned}$$

Cosine law



$$l_j = l_i \cos\theta_k + l_k \cos\theta_i$$

$$\begin{aligned}
2l_j l_k \cos\theta_i &= l_j^2 + l_k^2 - l_i^2 \\
2l_j &= 2l_k \cos\theta_i - 2l_j l_k \sin\theta_i \\
\frac{d\theta_i}{dl_j} &= \frac{l_k \cos\theta_i - l_j}{A} \\
&= -\frac{l_i \cos\theta_k}{A} \\
&= -\frac{d\theta_i}{dl_i} \cos\theta_k
\end{aligned}$$



$$l_k^2 = r_i^2 + r_j^2 + 2 r_i r_j l_{ij}$$

$$
\begin{aligned}
l_i^2 &= r_j^2 + r_k^2 + 2 r_j r_k l_{jk} \\
2 l_i \frac{dl_i}{dr_j} &= 2 r_j + 2 r_k l_{jk} \\
\frac{dl_i}{dr_j} &= \frac{2 r_j^2 + 2 r_j r_k l_{jk}}{2 l_i r_j} \\
&= \frac{r_j^2 + r_k^2 + 2 r_j r_k l_{jk} + r_j^2 - r_k^2}{2 l_i r_j} \\
&= \frac{l_i^2 + r_j^2 - r_k^2}{2 l_i r_j}
\end{aligned}
$$

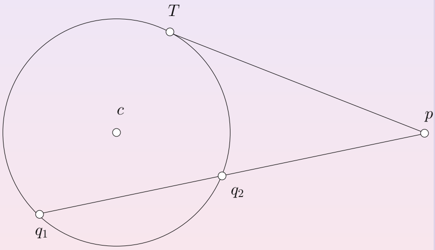
Cosine law

Let $u_i = \log r_i$, then

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A} \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix} \begin{pmatrix} -1 & \cos\theta_3 & \cos\theta_2 \\ \cos\theta_3 & -1 & \cos\theta_1 \\ \cos\theta_2 & \cos\theta_1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{l_1^2+r_2^2-r_3^2}{2l_1r_2} & \frac{l_1^2+r_3^2-r_2^2}{2l_1r_3} \\ \frac{l_2^2+r_1^2-r_3^2}{2l_2r_1} & 0 & \frac{l_2^2+r_3^2-r_1^2}{2l_2r_3} \\ \frac{l_3^2+r_1^2-r_2^2}{2l_3r_1} & \frac{l_3^2+r_2^2-r_1^2}{2l_3r_2} & 0 \end{pmatrix} \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}$$

Power

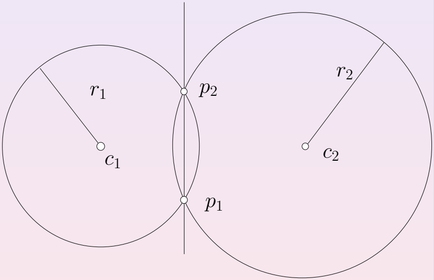


Power

Suppose a point p is not coincident of the center of a circle $\mathbf{c} = (c, r)$ on the plane, the line through c and p intersects the circle at q_1 and q_2, T is the tangent point, then the power of p with respect to the circle is

$$
\begin{aligned}
pow(p, \mathbf{c}) &= |pq_1||pq_2| \\
&= |pT|^2 \\
&= |pc|^2 - r^2.
\end{aligned}
$$

Power



Equi-Power line

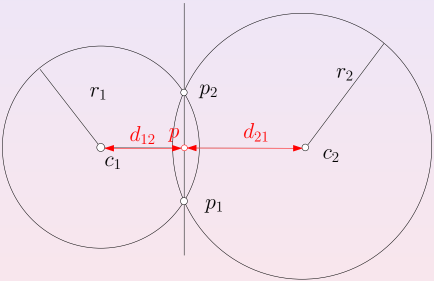
Suppose there are two circles $\mathbf{c_1} = (c_1, r_1)$, $\mathbf{c_2} = (c_2, r_2)$, the equi-power line is the locus

$$pow(p, \mathbf{c_1}) = pow(p, \mathbf{c_2}).$$

The equation of p is

$$|p - c_1|^2 - r_1^2 = |p - c_2|^2 - r_2^2.$$

If two circles intersect at p_1 and p_2, then the line through them is the equi-power line.



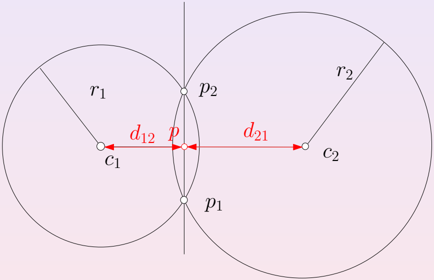
Suppose there are two circles $\mathbf{c_1} = (c_1, r_1)$, the line through c_1 and c_2 intersects the equi-power line at the point p. Assume the length between c_1 and c_2 is l. The distance from p to c_2 is denoted as d_{21}, then

$$d_{12} = \frac{l^2 + r_1^2 - r_2^2}{2l}$$

$$d_{21} = \frac{l^2 + r_2^2 - r_1^2}{2l}$$

obviously, $d_{12} + d_{21} = l$.

Power

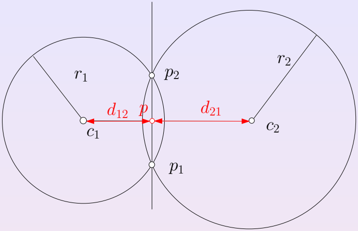


compute the power of p with respect to two circles

$$\begin{aligned} pow(p,\mathbf{c_1}) &= d_{12}^2 - r_1^2 \\ pow(p,\mathbf{c_2}) &= d_{21}^2 - r_2^2 \end{aligned}$$

$$\begin{aligned} d_{12}^2 - d_{21}^2 &= (d_{12}+d_{21})(d_{12}-d_{21}) \\ &= l\frac{r_1^2 - r_2^2}{l} = r_1^2 - r_2^2. \end{aligned}$$

Power



Lemma

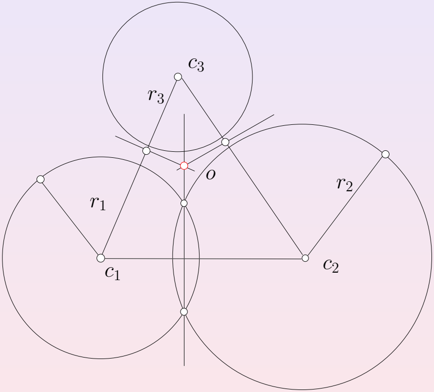
The equi-power line is orthogonal to the line connecting the centers.

Proof.

Define a function $\phi(p) = pow(p, \mathbf{c_1}) - pow(p, \mathbf{c_2})$,

$$\begin{aligned} \phi(p) &= \langle p - c_1, p - c_1 \rangle - r_1^2 - \langle p - c_2, p - c_2 \rangle + r_2^2 \\ d\phi(p) &= \langle dp, c_2 - c_1 \rangle \end{aligned}$$

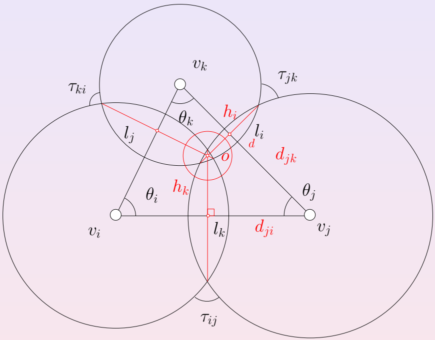
Power



Given three circles $\mathbf{c_k}, k = 1, 2, 3$, then three equi-power lines intersect at one point o, which is called the *power center*,
The equi-power lines of $\mathbf{c_1}, \mathbf{c_2}$ and $\mathbf{c_1}, \mathbf{c_3}$ intersects at the point o. Then

$$pow(o, \mathbf{c_1}) = pow(o, \mathbf{c}_2) = pow(o, \mathbf{c_3}$$

so o is also on the equi-power line of $\mathbf{c}_2, \mathbf{c_3}$.

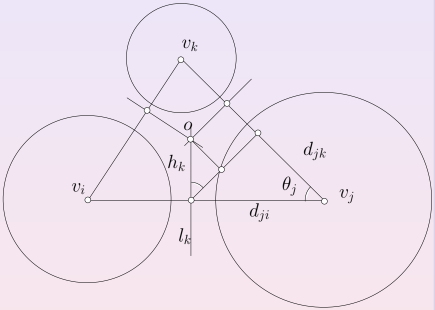


$$\frac{d\theta_i}{du_j} = \frac{h_k}{l_k}$$

There are 3 circles $\mathbf{c_k} = (c_k, r_k)$, the power center o is also the center of the unique circle (p, r), which is orthogonal to all 3 circles.

$$pow(p, \mathbf{c_k}) = \langle p - c_k, p - c_k \rangle - r_k^2 = r^2$$

so the power center is the center of the circle which is orthogonal to the 3 circles.



$$l_k^2 = r_i^2 + r_j^2 + 2 l_{ij} r_i r_j$$

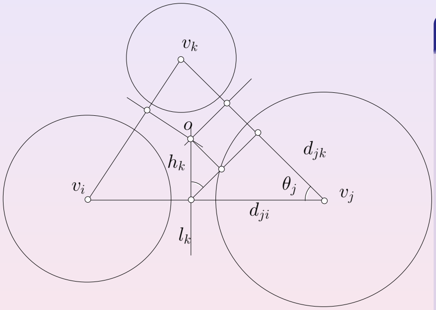
$$2l_k \frac{dl_k}{dr_j} = 2r_j + 2r_i l_{ij}$$

$$r_j \frac{dl_k}{dr_j} = \frac{2r_j^2 + 2r_i r_j l_{ij}}{2l_k}$$

$$= \frac{l_k^2 + r_j^2 - r_i^2}{2l_k}$$

Therefore

$$\frac{dl_k}{du_j} = d_{ji}.$$

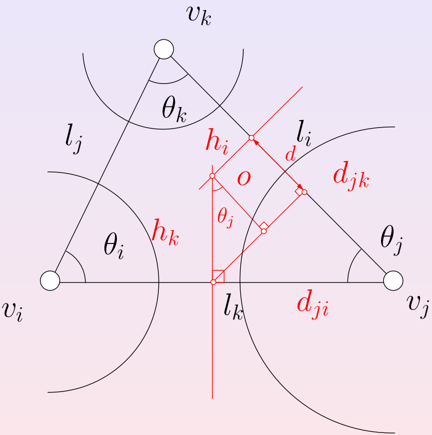


The distance from o to edge $[v_i, v_j]$ is h_k.

Theorem

$$\frac{d\theta_i}{du_j} = \frac{d\theta_j}{du_i} = \frac{h_k}{l_k}$$
$$\frac{d\theta_j}{du_k} = \frac{d\theta_k}{du_j} = \frac{h_i}{l_i}$$
$$\frac{d\theta_k}{du_i} = \frac{d\theta_i}{du_k} = \frac{h_j}{l_j}$$

Cosine law



$$\frac{d\theta_i}{du_j} = \frac{h_k}{l_k}$$

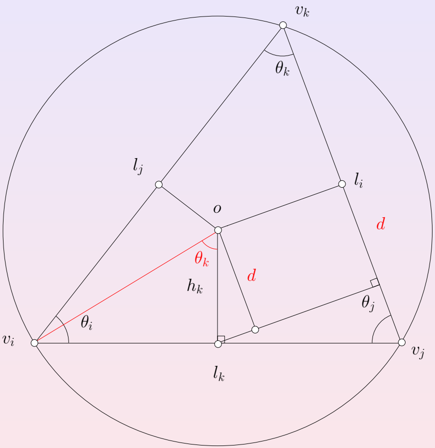
Proof.

$$\begin{aligned}
\frac{\partial \theta_i}{\partial u_j} &= \frac{\partial \theta_i}{\partial l_i}\frac{\partial l_i}{\partial u_j} + \frac{\partial \theta_i}{\partial l_k}\frac{\partial l_k}{\partial u_j} \\
&= \frac{\partial \theta_i}{\partial l_i}\left(\frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j}\cos\theta_j\right) \\
&= \frac{l_i}{A}(d_{jk} - d_{ji}\cos\theta_j) \\
&= \frac{dl_i}{l_i l_k \sin\theta_j} \\
&= \frac{h_k \sin\theta_j}{l_k \sin\theta_j} \\
&= \frac{h_k}{l_k}
\end{aligned}$$

□

Inversive Distance CP Metric - Local Rigidity

The Discrete Ricci energy of Inversive distance CP metric is convex, but the metric space is non-convex. Therefore it has local rigidity, not global rigidity.



$$l_{ij} \leftarrow e^{u_i} l_{ij} e^{u_j}$$

Shrink three circles to vertices, then the power center o becomes the circum-center.

$$\begin{aligned}
\frac{\partial \theta_i}{\partial u_j} &= \frac{\partial \theta_i}{\partial l_i}\left(\frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j}\cos\theta_j\right) \\
&= \frac{l_i}{A}(l_i - l_k \cos\theta_j) \\
&= \frac{2 l_i d}{l_i l_k \sin\theta_j} \\
&= \frac{2h_k}{l_k} \\
&= \cot\theta_k
\end{aligned}$$

Discrete Yamabe flow - Local Rigidity

The Discrete Ricci energy of discrete Yamabe flow is convex, but the metric space is non-convex. Therefore it has local rigidity, not global rigidity.