Hyperbolic Discrete Ricci Curvature Flows

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Unified Framework of Discrete Curvature Flow

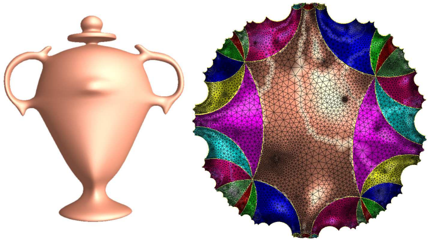
Unified framework for both Discrete Ricci flow and Yamabe flow

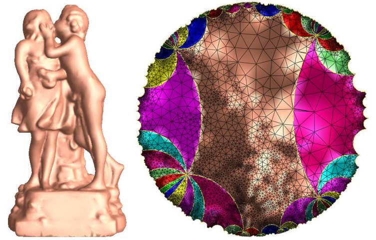
- Curvature flow
$$\frac{du}{dt} = \bar{K} - K,$$
- Energy
$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$
- Hessian of E denoted as Δ,
$$d\mathbf{K} = \Delta d\mathbf{u}.$$

Newton's method.

Hyperbolic Ricci Flow

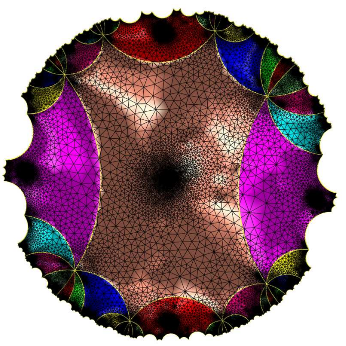
Computational results for genus 2 and genus 3 surfaces.



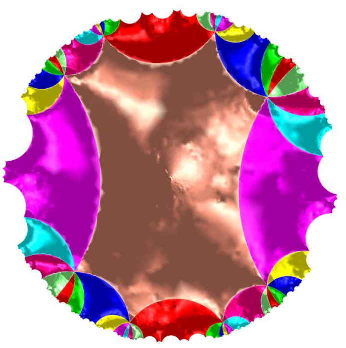


Hyperbolic Ricci flow

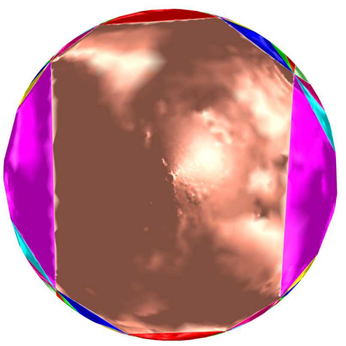
Hyperbolic Ricci flow



Hyperbolic Ricci flow



Poincaré Model



Klein Model

Hyperbolic Functions

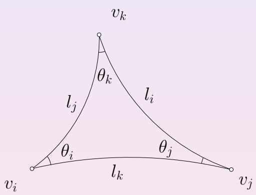
$$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \coth x = \frac{\cosh x}{\sinh x}$$

$$\sinh' = \cosh, \cosh' = \sinh,$$

$$\tanh' = \frac{1}{\cosh^2}, \coth' = -\frac{1}{\sinh^2}$$

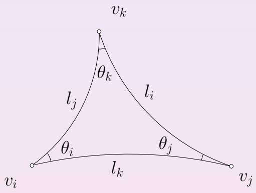
$$\cosh^2 - \sinh^2 = 1.$$



$$\cos\theta_i = \frac{\cosh l_j \cosh l_k - \cosh l_i}{\sinh l_j \sinh l_k}$$

$$\frac{\sin\theta_i}{\sinh l_i} = \frac{\sin\theta_j}{\sinh l_j} = \frac{\sin\theta_k}{\sinh l_k}$$

Cosine law



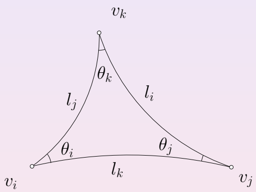
$$A = \sinh l_j \sinh l_k \sin \theta_i$$

$$\cos \theta_i = \frac{\cosh l_j \cosh l_k - \cosh l_i}{\sinh l_j \sinh l_k}$$

$$-\sinh l_i = -\sinh l_j \sinh l_k \sin \theta_i \frac{\partial \theta_i}{\partial l_i}$$

$$\frac{\partial \theta_i}{\partial l_i} = \frac{\sinh l_i}{A}$$

Cosine law



$$A = \sinh l_j \sinh l_k \sin\theta_i$$

$$\begin{aligned}
\cos\theta_i &= \frac{\cosh l_j \cosh l_k - \cosh l_i}{\sinh l_j \sinh l_k} \\
\frac{\partial\theta_i}{\partial l_j} &= \frac{S_j C_k (S_j S_k) - (-C_i + C_j C_k) S_k C_j}{-\sin\theta_i (S_j S_k)^2} \\
&= \frac{(S_j^2 - C_j^2) C_k S_k + C_i C_j S_k}{-\sin\theta_i S_j^2 S_k^2} \\
&= \frac{-C_k + C_i C_j}{-A S_j} \\
&= \frac{S_i S_j \cos\theta_k}{-A S_j} \\
&= -\frac{S_i}{A}\cos\theta_k
\end{aligned}$$

Cosine law

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A} \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix} \begin{pmatrix} -1 & \cos\theta_3 & \cos\theta_2 \\ \cos\theta_3 & -1 & \cos\theta_1 \\ \cos\theta_2 & \cos\theta_1 & -1 \end{pmatrix} \begin{pmatrix} dl_1 \\ dl_2 \\ dl_3 \end{pmatrix}$$

Hyperbolic cosine law

$$\cosh l_i = \cosh r_j \cosh r_k + \sinh r_j \sinh r_k l_{jk}$$

$$\sinh l_i \frac{dl_i}{dr_j} = \sinh r_j \cosh r_k + l_{jk} \cosh r_j \sinh r_k$$

$$\begin{aligned}
\frac{dl_i}{dr_j} &= \frac{\sinh r_j \cosh r_k + \cosh r_j \sinh r_k l_{jk}}{\sinh l_i} \\
l_{jk} &= \frac{\cosh l_i - \cosh r_j \cosh r_k}{\sinh r_j \sinh r_k}
\end{aligned}$$

$$
\begin{aligned}
\frac{dl_i}{dr_j} &= \frac{\sinh r_j \cosh r_k + \cosh r_j \sinh r_k \frac{\cosh l_i - \cosh r_j \cosh r_k}{\sinh r_j \sinh r_k}}{\sinh l_i} \\
&= \frac{\sinh^2 r_j \cosh r_k + \cosh r_j \cosh l_i - \cosh^2 r_j \cosh r_k}{\sinh l_i \sinh r_j} \\
&= \frac{(\sinh^2 r_j - \cosh^2 r_j)\cosh r_k + \cosh r_j \cosh l_i}{\sinh l_i \sinh r_j} \\
&= \frac{\cosh r_j \cosh l_i - \cosh r_k}{\sinh l_i \sinh r_j}
\end{aligned}
$$

In Euclidean case

$$\frac{dl_i}{dr_j} = \frac{r_j^2 + l_i^2 - r_k^2}{l_i r_j}$$

dual

$$l_i \to \sinh l_i, \cosh r_j \to r_j^2.$$

hyperbolic cosine law

$$u = \log \tanh \frac{r}{2}$$

$$\begin{aligned}
\frac{du}{dr} &= \frac{1}{\tanh \frac{r}{2}} \frac{1}{\cosh^2 \frac{r}{2}} \frac{1}{2} \\
&= \frac{1}{2 \sinh \frac{1}{2} \cosh \frac{r}{2}} \\
&= \frac{1}{\sinh r}
\end{aligned}$$

Hyperbolic cosine law

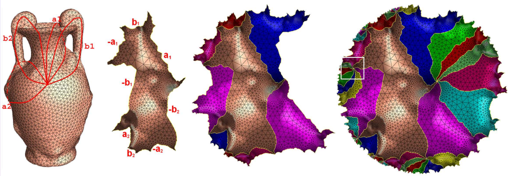
$$\begin{pmatrix} dl_1 \\ dl_2 \\ dl_3 \end{pmatrix} = MD \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & \frac{-\cosh r_3 + \cosh l_1 \cosh r_2}{\sinh l_1 \sinh r_2} & \frac{-\cosh r_2 + \cosh l_1 \cosh r_3}{\sinh l_1 \sinh r_3} \\ \frac{-\cosh r_3 + \cosh l_2 \cosh r_1}{\sinh l_2 \sinh r_1} & 0 & \frac{-\cosh r_1 + \cosh l_2 \cosh r_3}{\sinh l_2 \sinh r_3} \\ \frac{-\cosh r_2 + \cosh l_3 \cosh r_1}{\sinh l_3 \sinh r_1} & \frac{-\cosh r_1 + \cosh l_3 \cosh r_2}{\sinh l_3 \sinh r_2} & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} \sinh r_1 & 0 & 0 \\ 0 & \sinh r_2 & 0 \\ 0 & 0 & \sinh r_3 \end{pmatrix}$$

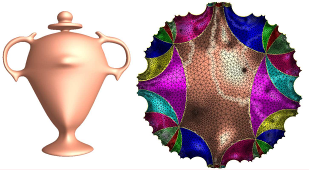
Compute Teichmüller coordinates

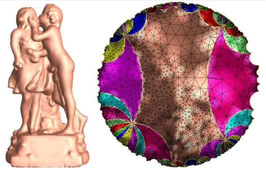
Step 1. Compute the hyperbolic uniformization metric.



(a) (b) (c) (d)

Step 2. Compute the Fuchsian group generators.





Teichmüller space

Theorem (Teichmüller space)

The Teichmüller space of a genus g closed surface is of $6g-6$ dimension.

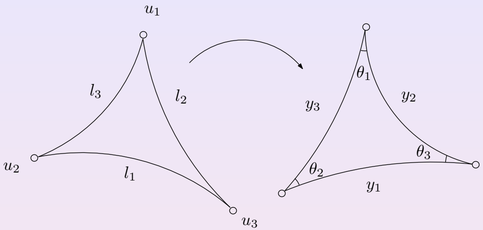
Proof.

Suppose $\{a_1,b_1,\cdots,a_g,b_g\}$ are the generators of $\pi_1(M)$, then the corresponding deck transformations are $\{\alpha_1,\beta_1,\cdots,\alpha_g,\beta_g\}$. Under the hyperbolic uniformization metric, each α_k and β_k are Möbius transformations, therefore require 3 parameters, and

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}=id,$$

therefore β_g can be determined from other generators. Therefore, there are total $3(g-1)$ parameters. On the other hand, the dimension of hyperbolic isometries is 3, so the dimension of Teichmüller space of a genus g surface is $6g-6$. □

Discrete conformal metric deformation:



conformal factor

$$
\begin{aligned}
\frac{y_k}{2} &= e^{u_i}\frac{l_k}{2}e^{u_j} && \mathbb{R}^2\\
\sinh\frac{y_k}{2} &= e^{u_i}\sinh\frac{l_k}{2}e^{u_j} && \mathbb{H}^2\\
\sin\frac{y_k}{2} &= e^{u_i}\sin\frac{l_k}{2}e^{u_j} && \mathbb{S}^2
\end{aligned}
$$

Properties: $\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$ and $d\mathbf{K} = \Delta d\mathbf{u}$.

Hyperbolic Yamabe Flow

$$
\begin{aligned}
\sinh\frac{y_k}{2} &= e^{u_i}\sinh\frac{l_k}{2}e^{u_j}\\
\frac{1}{2}\cosh\frac{y_k}{2}\frac{\partial y_k}{\partial u_i} &= u_i e^{u_i}\sinh\frac{l_k}{2}e^{u_j}\\
&= u_i\sinh\frac{y_k}{2}\\
\frac{\partial y_k}{\partial u_i} &= 2u_i\tanh\frac{y_k}{2}
\end{aligned}
$$

$$\begin{pmatrix} dy_1 \\ dy_2 \\ dy_3 \end{pmatrix} = \begin{pmatrix} \tanh\frac{y_1}{2} & 0 & 0 \\ 0 & \tanh\frac{y_2}{2} & 0 \\ 0 & 0 & \tanh\frac{y_3}{2} \end{pmatrix} \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}$$

Maximal hyperbolic Ricci flow

Discrete Riemann mapping

Let Ω be a simply connected domain contained in the unit disk. Compute a triangulation and construct a mesh. Run hyperbolic ricci flow, such that

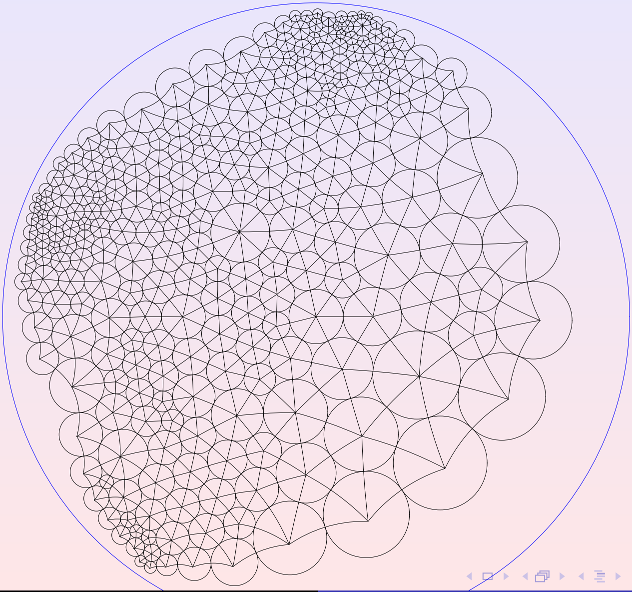
$$\forall v_i \in \partial M, u_i \to \infty.$$

Then the resulting mapping is the discrete Riemann mapping.

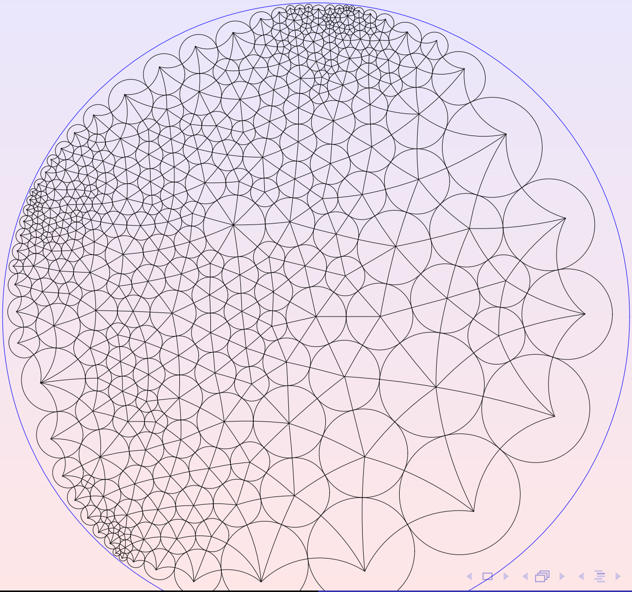
Theorem

Discrete Riemann mappingConformal mapping from a simply connected planar domain to the unit disk exists, and unique up to a rigid motion.

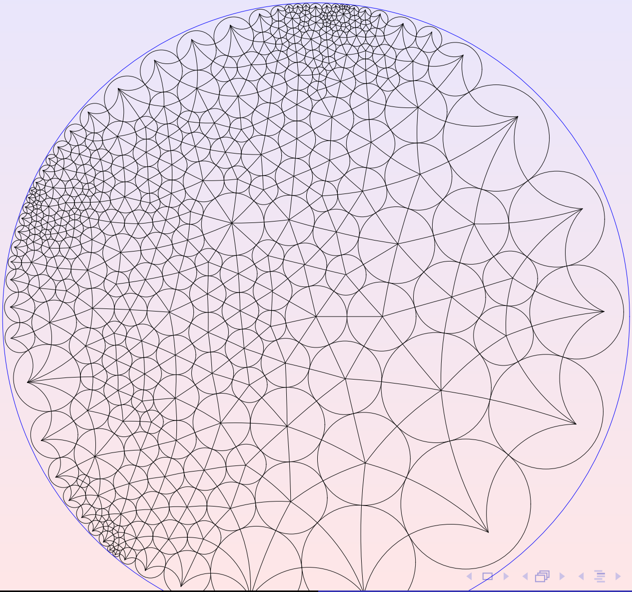
Maximal Hyperbolic Ricci flow



Maximal Hyperbolic Ricci flow



Maximal Hyperbolic Ricci flow



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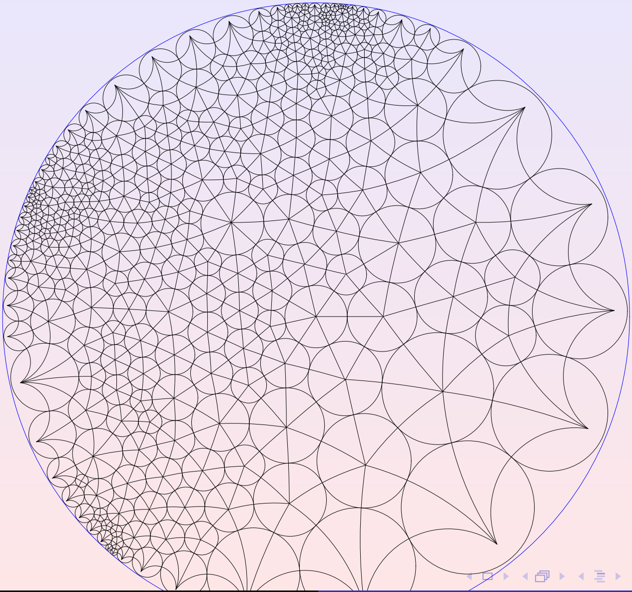
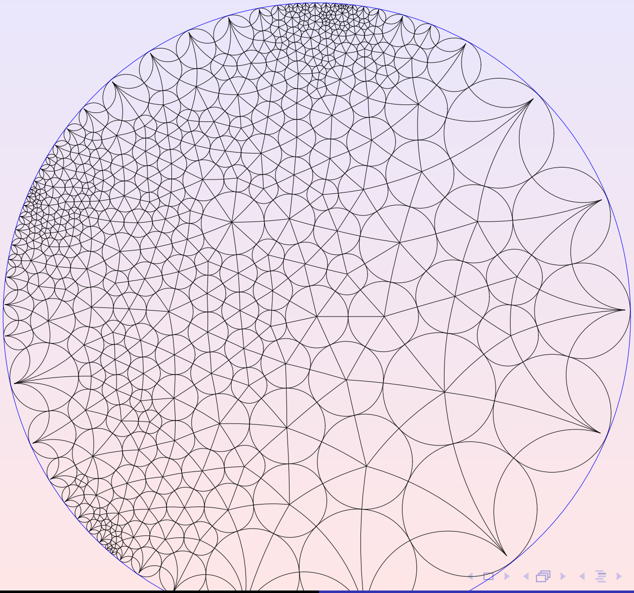


Maximal Hyperbolic Ricci flow



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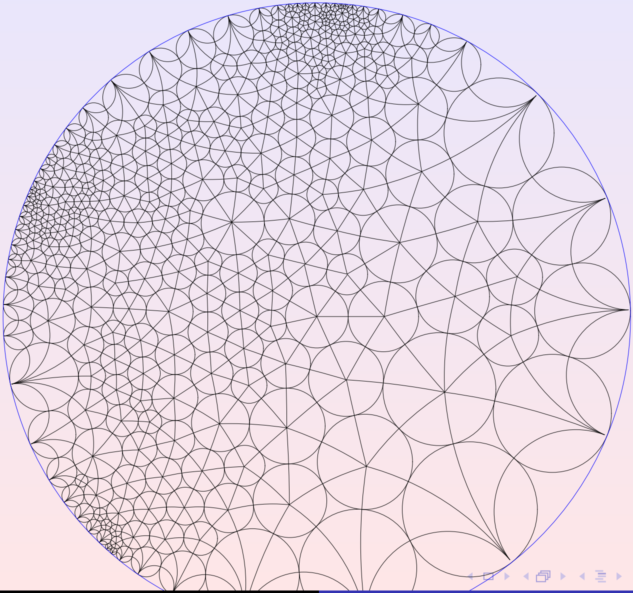
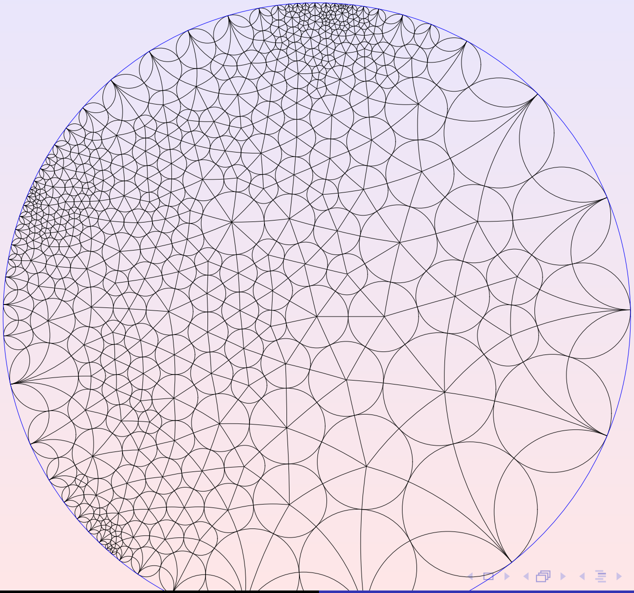


Maximal Hyperbolic Ricci flow



Maximal Hyperbolic Ricci flow