Graph Drawing

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Embedding Planar Graph

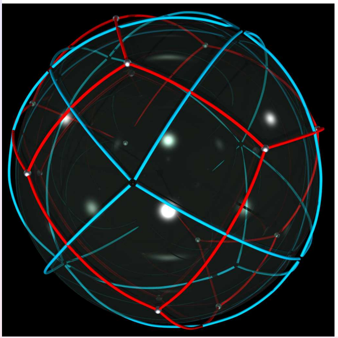


Figure: Embedding a planar graph (genus 0) onto the sphere.

Embedding Planar Graph

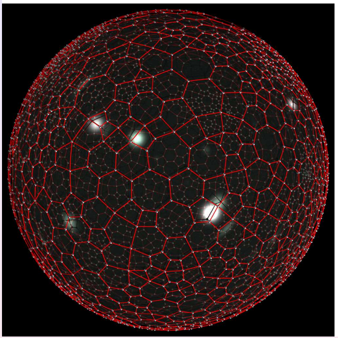
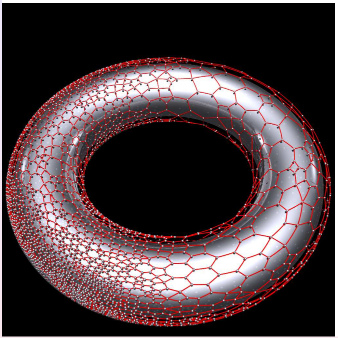
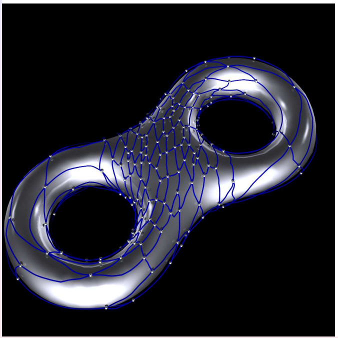


Figure: Embedding a planar graph (genus 0) onto the sphere.

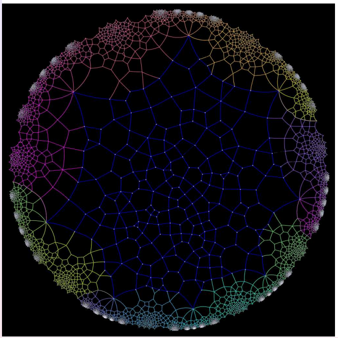
Embedding Genus One



Embedding Genus Two



Embedding Genus Two



Graph Drawing

1. Compute Ribbon graph of the graph G.
2. Compute the dual graph $\bar{G}$.
3. compute a constant curvature metric on the complex $G \cup \bar{G}$.

Ribbon Graph

1. Define a cyclic order of all edges adjacent to one node.
2. Associate two opposite half edges for each edge.
3. Trace along half edges to form a face. exhaust all half edges.

We obtain an oriented surface, the Ribbon graph of the input graph.

Dual Graph

1. Each node corresponds to a face.
2. Each face corresponds to a node.
3. Each edge corresponds to an edge.

The overlap of the graph and its dual graph is a oriented surface with quadrilateral faces.

Ricci flow

Add diagonal connecting the node vertex and the face vertex, to triangulate the complex.

1. Each node vertex is associated with a circle.
2. Each face vertex is associated with a circle.
3. Each edge vertex is associated with a zero radius circle.
4. All circle intersection angles are right angles.

Ricci flow gives constant curvature metric.

Graph Embedding

Theorem (Graph Embedding)

Any 3-connected graph G can be embedded onto an oriented surface.

Computing the minimal genus of the embedding surface is NP hard.

Embedding Planar Graph

We select one edge node to be mapped to the infinity point. We call it the *infinity edge node* and denote it as e_∞. The choice of the edge could be arbitrary. The edge node $e_\infty = [v_1, v_2] \cap [f_1, f_8]$ is selected as the infinity edge node. Suppose the infinity edge node is given by $e_\infty = [v_i, v_j] \cap [f_k, f_l]$, then we remove all the quadrilateral facets adjacent to v_i, v_j or f_k, f_l in the overlapping graph D, to get the reduced graph $\bar{G}$.

There are four quadrilateral adjacent to e_∞, each quadrilateral has a unique edge node other than e_∞. Denote them as $\{e_1, e_2, e_3, e_4\}$. The target curvature is set to be $\frac{\pi}{2}$ for these four nodes, and zero everywhere else, including the boundary nodes.

Embedding Planar Graph

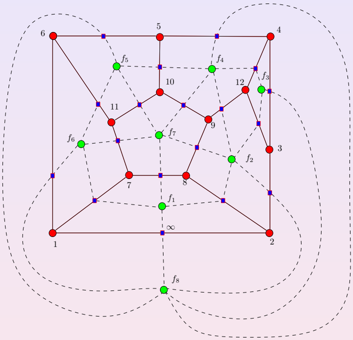
For genus zero graph, we flatten its reduced graph onto the plane. We choose a root face, and embed it onto the plane isometrically. The put its neighbors into a queue. When the queue is not empty, we pop the first face, $[v_i, v_j, v_k]$. Two vertices of the face must have been embedded already, assume they are v_i and v_j. Then v_k is the intersection of two circles (v_i, l_{ik}) and (v_j, l_{jk}) and the normal of v_i, v_j, v_k is consistent with that of the plane. After embedding the head face, we check all the faces sharing an edge with it, and if they haven't been embedded, put them into the queue. By repeating this procedure, we flatten face by face, and eventually flatten the whole mesh onto the plane. The planar image is a rectangle.

Planar Graph Embedding

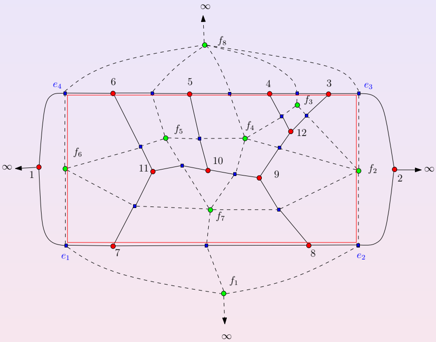
For genus zero graph G, we embed the reduced graph $\bar{G}$ to a planar rectangle, then we use stereographic projection to map the whole plane to the unit sphere $\mathbb{S}^2$, $\tau : (u,v) \to (x,y,z)$

$$\tau(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right).$$

Stereographic projection maps planar circles and lines to spherical circles. In Figure 4, suppose $e_\infty = [v_i, v_j] \cap [f_k, f_l]$, the vertical lines of the rectangle are mapped to the vertex node circles of v_i and v_j, the horizontal lines correspond to the face node circles of f_k and f_l.



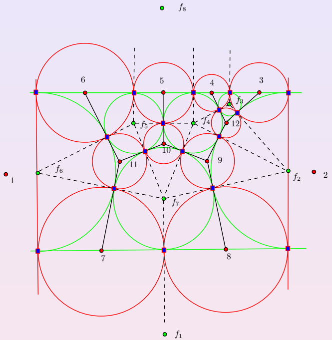
(a) Input graph and its dual



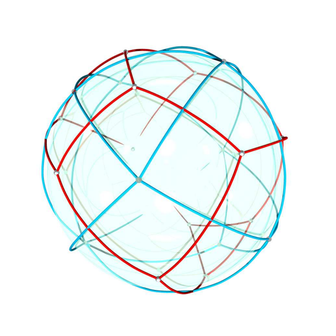
(b) The reduced graph

Figure: Embedding a planar graph (genus 0) onto the sphere.

Planar Graph Embedding



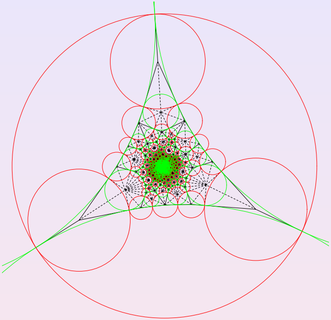
(c) Ricci flow result



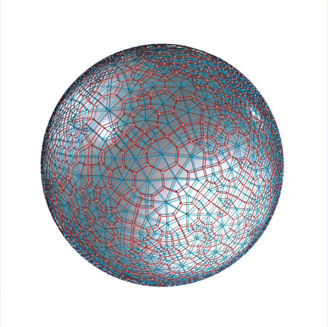
(d) Spherical embedding.

Figure: Embedding a planar graph (genus 0) onto the sphere.

Planar Graph Embedding

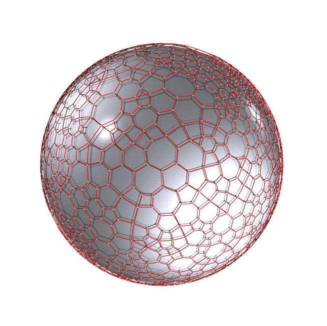


(a) Graph embedding on complex sphere;

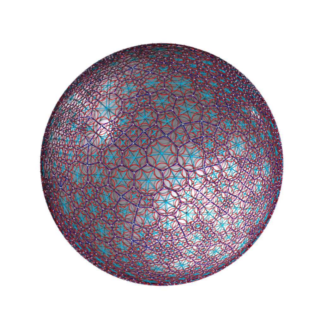


(b) Overlapped Graph embedding on the sphere in R^3.

Figure: Genus zero mesh graph embedding



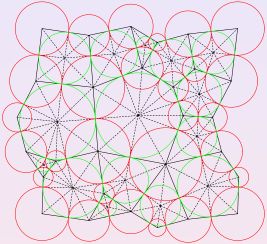
(c) Graph embedding on the sphere;



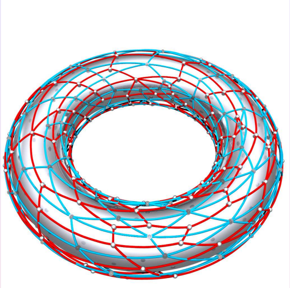
(d) Graph embedding with circle packing.

Figure: Genus zero mesh graph embedding

Genus one Graph Embedding



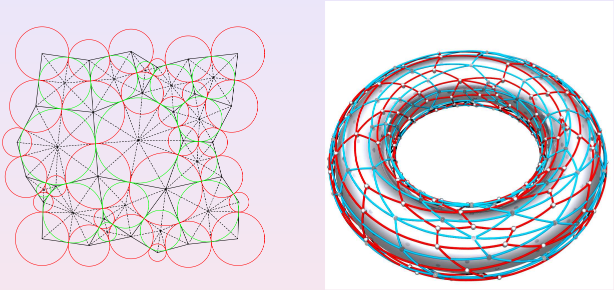
(a) Flatten on $\mathbb{R}^2$



(b) Embed on a surface

Figure: Embedding a genus one graph onto a torus.

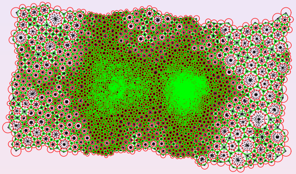
Genus one Graph Embedding



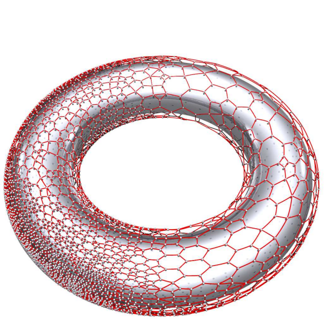
(a) Flatten on $\mathbb{R}^2$ (b) Embed on a surface

Figure: Embedding a genus one graph onto a torus.

Genus One Graph Embedding

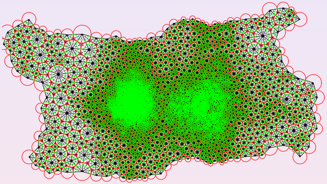


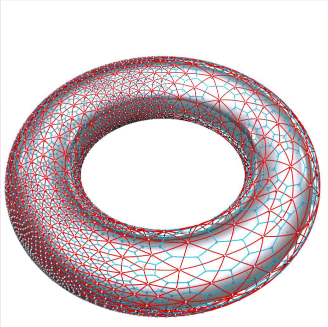
(a) Graph embedding on the plane;



(b) Graph embedding in R^3.

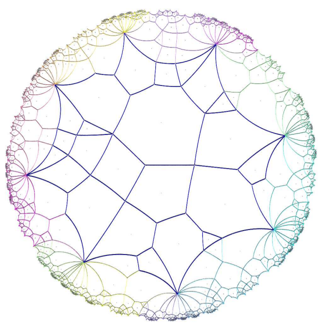
Genus One Graph Embedding





(a) Graph embedding on the plane; (b) Overlapped graph embedding.

Genus 2 Graph Embedding

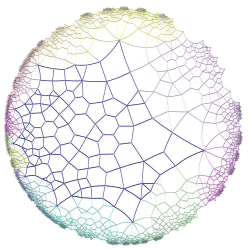


(a) Genus 2 graph embedding on $\mathbb{H}^2$;

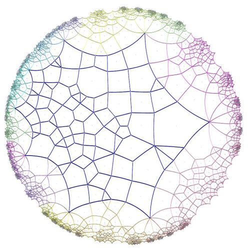
(b) Genus 2 graph embedding in R^3.

Figure: Genus 2 graph embedding on Riemann surfaces

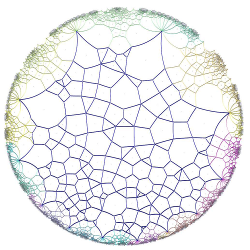
Genus 3 Graph



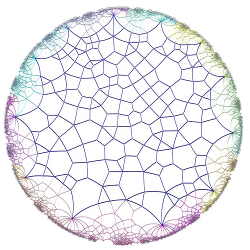
(a) Genus 2 graph;



(b) Genus 2 graph.



(c) Genus 3 graph;



(d) Genus 3 graph;

Figure: Genus 3 graphs embedding on Riemann surfaces