# Weierstrass Representation

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# Weierstrass Representation

Suppose $\pi$ is a plane through the origin, namely a two dimensional linear subspace in $\mathbb{R}^3$. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors in $\pi$, such that $\mathbf{u} \perp \mathbf{v}$, $|\mathbf{u}| = |\mathbf{v}|$. Let $y_k = u_k + iv_k$, $k = 1, 2, 3$, then

$$y_1^2 + y_2^2 + y_3^2 = 0.$$

Let $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ are another set of orthogonal vectors on $\pi$, with the same length. $(\mathbf{u},\mathbf{v})$ rotates by angle $\phi$ to $(\tilde{\mathbf{u}},\tilde{\mathbf{v}})$, then

$$\tilde{y}_k = re^{i\phi} y_k.$$

so each plane has a unique representation $(y_1 : y_2 : y_3) = (\tilde{y}_1 : \tilde{y}_2 : \tilde{y}_3) \in \mathbb{CP}^2$, such that

$$y_1^2 + y_2^2 + y_3^2 = \tilde{y}_1^2 + \tilde{y}_2^2 + \tilde{y}_3^2 = 0.$$

# Weierstrass Representation

Grassman manifold $G_{3,2}$ has representation

$$G_{3,2} = \{(y_1, y_2, y_3) \in \mathbb{CP}^2 | y_1^2 + y_2^2 + y_3^2 = 0\}$$

# Weierstrass Representation

$G_{3,2}$ has rational parameterization $\mathbb{CP}^1$, $(a:b) \in \mathbb{CP}^1$,

$$\begin{cases} y_1 &= \frac{i}{2}(b^2 + a^2) \\ y_2 &= \frac{1}{2}(b^2 - a^2) \\ y_3 &= ab \end{cases}$$

# Surface

Suppose $\mathbf{r}(u,v) = (x^1(u,v), x^2(u,v), x^3(u,v))$ is a surface immersed in $\mathbb{R}^3$, $(u,v)$ are conformal parameterization, isothermal coordinates

$$\mathbf{r}_u := \frac{\partial \mathbf{r}}{\partial u} = (x_u^1, x_u^2, x_u^3);$$

and

$$\mathbf{r}_v := \frac{\partial \mathbf{r}}{\partial v} = (x_v^1, x_v^2, x_v^3);$$

then

$$\begin{aligned} \langle \mathbf{r}_u, \mathbf{r}_u \rangle &= \langle \mathbf{r}_v, \mathbf{r}_v \rangle \\ \langle \mathbf{r}_u, \mathbf{r}_v \rangle &= 0 \end{aligned}$$

# Complex Differential Operator

Complex differential operators are

$$dz = dx + idy, d\bar{z} = dx - idy,$$

the dual differential operator

$$\langle \frac{\partial}{\partial z}, dz \rangle = 1, \langle \frac{\partial}{\partial \bar{z}}, d\bar{z} \rangle = 1,$$

we get

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

The tangent plane at $\mathbf{r}(u,v)$ is the $\pi := Span\{\mathbf{r}_u, \mathbf{r}_v\}$.

$$\pi = (x_u^1 + ix_v^1, x_u^2 + ix_v^2, x_u^3 + ix_v^3) = (x_z^1, x_z^2, x_z^3),$$

Then the tangent plane in $G_{3,2}$ is given by

$$(x_z^1)^2 + (x_z^2)^2 + (x_z^3)^2 = 0.$$

Then we use rational parameter $(a:b) \in \mathbb{CP}^1$ for $G_{3,2}$

$$\begin{cases} x_z^1 &= \frac{i}{2}(b^2 + a^2) \\ x_z^2 &= \frac{1}{2}(b^2 - a^2) \\ x_z^3 &= ab \end{cases}$$

Let

$$\begin{cases} \psi_1 & = & a \\ \psi_2 & = & \bar{b} \end{cases}$$

Then

$$\begin{cases} x_z^1 & = & \frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2) \\ x_z^2 & = & \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2) \\ x_z^3 & = & \psi_1 \bar{\psi}_2 \end{cases}$$

The Riemannian metric

$$ds^2 = e^{2\alpha} dz d\bar{z}, e^{\alpha} = |\psi_1|^2 + |\psi_2|^2$$

The mean curvature $H$

$$\Delta \mathbf{r} = 2H\mathbf{n},$$

where

$$\Delta = 4e^{-2\alpha}\partial\bar{\partial} = 4e^{-2\alpha}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}.$$

The Gaussian curvature is given by

$$K = -4e^{-2\alpha}\alpha_{z\bar{z}} = -\Delta\alpha$$

# Dirac Equation

## Potential Function

$$U = \frac{He^{\alpha}}{2}$$

Because $x^k \in \mathbb{R}$ then

$$Img(x^k_{z\bar{z}}) = 0,$$

we get Dirac equation for $\psi := (\psi_1, \psi_2)$

$$D\psi = 0,$$

where $D$ operator is given by

$$D = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

which is the square root of Schrödinger operator with potential field $U$

$$L = \partial\bar{\partial} - \frac{\partial U}{U}\bar{\partial} + U^2$$

# Weierstrass Representation

$$\Phi = x^2 + ix^1$$

$$\psi_1 = \sqrt{-\Phi_z}, \psi_2 = \sqrt{\Phi_{\bar{z}}}$$

$$\begin{cases} x^1 & = & \frac{i}{2}\int(\bar{\psi}_2^2 + \psi_1^2)dz - (\bar{\psi}_1^2 + \psi_2^2)d\bar{z} \\ x^2 & = & \frac{1}{2}\int(\bar{\psi}_2^2 - \psi_1^2)dz - (\bar{\psi}_1^2 - \psi_2^2)d\bar{z} \\ x^3 & = & \int \psi_1\bar{\psi}_2 dz + \bar{\psi}_1\psi_2 d\bar{z} \end{cases}$$

# Global Weierstrass Representation

Torus case

$$(\psi_1, \bar{\psi}_2) \to \pm(\psi_1, \bar{\psi}_2), U \to U.$$

High genus case, using upper half plane model, Fuchsian transformation is

$$z \to \gamma(z) = \frac{az+b}{cz+d}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

then

$$\begin{cases} (\psi_1, \bar{\psi}_2) & \to & (cz+d)(\psi_1, \bar{\psi}_2) \\ U & \to & |cz+d|^2 U \end{cases}$$

If the global Weierstrass representation of the immersion of the universal covering space of $\Sigma_0$ is a closed surface, then for any holomorphic differentials on $\Sigma_0$

$$\int_{\Sigma_0} \bar{\psi}_1^2 d\bar{z} \wedge \omega = \int_{\Sigma_0} \psi_2^2 d\bar{z} \wedge \omega = \int_{\Sigma_0} \bar{\psi}_1 \psi_2 d\bar{z} \wedge \omega = 0.$$