Möbius Transformation

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Rigidity

Conformal mappings have rigidity. The diffeomorphism group is of infinite dimension in general. Conformal diffeomorphism group is of finite dimension. By fixing the topology, or several points, we can fix the entire conformal mapping.

Möbius Transformation

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Definition (Möbius Transformation)

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation, if it has the format

$$\phi(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad - bc = 1.$$

Möbius Transformation

Theorem

All Möbius transformations form a group.

We use complex homogenous coordinates to represent the Riemann sphere $z = (zw, w), w \in \mathbb{C}$, then Möbius transformation has the matrix representation

$$\phi(z) \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

therefore the set of all Möbius transformation is equivalent to the matrix group $SL(\mathbb{C}, 2)$.

Lemma

A Möbius transformation is the composition of translation, inversion, reflection rotation, and dilation.

$$f_1(z) = z + \frac{d}{z}, f_2(z) = \frac{1}{z}, f_3(z) = -\frac{ad-bc}{c^2}z, f_4(z) = z + \frac{a}{c},$$

by direct computation

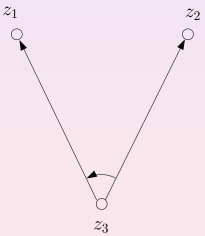
$$f_4 \circ f_3 \circ f_2 \circ f_1(z) = f(z) = \frac{az+b}{cz+d}.$$

Ratio

Definition (Ratio)

The ratio of three points on the Riemann sphere is given by

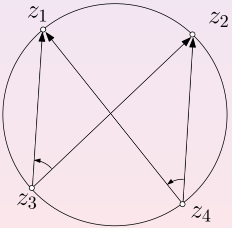
$$[z_1, z_2; z_3] = \frac{z_1 - z_3}{/} \frac{z_2 - z_3}{.}$$



Definition (Cross Ratio)

The cross ratio of four points on the Riemann sphere is given by

$$[z_1, z_2; z_3, z_4] = \frac{[z_1, z_2; z_3]}{[z_1, z_2; z_4]} = \frac{z_1 - z_3}{z_1 - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4}.$$



Theorem

Möbius transformations preserve cross ratios.

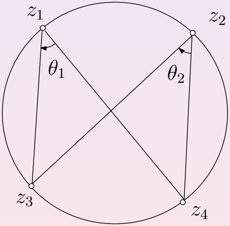
By definition, it is obvious that translation, rotation, dilation preserve cross ratio. It is sufficient to show that $\frac{1}{z}$ preserves cross ratio.

$$\frac{\frac{1}{z_1}-\frac{1}{z_3}}{\frac{1}{z_1}-\frac{1}{z_4}}\frac{\frac{1}{z_2}-\frac{1}{z_4}}{\frac{1}{z_2}-\frac{1}{z_3}}=\frac{z_3-z_1}{z_4-z_1}\frac{z_4-z_2}{z_3-z_2}.$$

Corollary

Möbius transformations preserve circles.

Four points are on a circle, if and only if $[z_1, z_2; z_3, z_4]$ is a real number.



Spherical Conformal Mapping

Theorem (Spherical Conformal Automorphism)

Suppose $f: \mathbb{S}^2 \to \mathbb{S}^2$ is a biholomorphic automorphism, then f must be a linear rational function.

The sphere is the Riemann sphere $\mathbb{C} \cup \{\infty\}$. First, the poles of f must be finite. Suppose there are infinite poles of f, because $\mathbb{S}^2$ is compact, there must be accumulation points, then f must be a constant value function, contradiction to the fact that f is an automorphism. Let $z_1, z_2, \cdots, z_n$ be the finite poles of f, with degrees $e_1, e_2, \cdots, e_n$.

Let $g = \pi_i (z - z_i)_i^e$, then fg is a holomorphic function on $\mathbb{C}$, therefore fg is entire, namely, fg is a polynomial. Therefore

$$f = \frac{\sum_i^n a_k z^i}{\sum_j^m b_j z^j},$$

if $n > 1$ then f has multiple zeros, contradict to the condition that f is an automorphism, therefore $n = 1$. Similarly, $m = 1$.

After normalization, spherical harmonic maps are Möbius transformations. Let $\{z_0, z_1, z_2\}$ be three distinct points on the complex plane, the unique Möbius map which maps them to $\{0, 1, \infty\}$ is

$$z \to \frac{z - z_0}{z - z_2}\frac{z_1 - z_2}{z_1 - z_0}.$$

UHP conformal mapping

Definition (Upper Half Plane)

The UHP is the upper half plane in $\mathbb{C}$, $\{z \in \mathbb{C} | img(z) > 0\}$.

Theorem (UHP Conformal Mapping)

Suppose $f : UHP \to UHP$ is a conformal automorphism, then $f \in SL(R, 2)$.

We analytically extend f to a conformal automorphism of the Riemann sphere, $\tilde{f} : \mathbb{S}^2 \to \mathbb{S}^2$

$$\tilde{f}(z) = \begin{cases} f(z) & img(z) \geq 0 \\ \overline{f(\bar{z})} & img(z) < 0 \end{cases}$$

Then $\tilde{f} \in SL(\mathbb{C}, 2)$. Because $\tilde{f}(z) = \overline{\tilde{f}(\bar{z})}$, the symmetry ensures $f \in SL(\mathbb{R}, 2)$.

Disk Conformal Mapping

Theorem (Disk Conformal Automorphism)

Suppose $f:\mathbb{D}\to\mathbb{D}$ is a conformal automorphism from the unit disk to itself, then f has the form

$$z \to e^{i\theta}\frac{z-z_0}{1-\bar{z}_0 z}.$$

Find three points on the unit disk $\{a,b,c\}$, construct a Möbius transformation $\phi:\mathbb{D}\to UHP$, which maps them to $\{0,1,\infty\}$. Then $\phi\circ f\circ\phi^{-1}:UHP\to UHP$ is a conformal automorphism of UHP, which belongs to $SL(\mathbb{R},2)$. By direct computation, the above can be shown.

fixed point and fixed circles

Definition (fixed point)

Let f be a Möbius transformation, $z_0 \in \mathbb{C} \cup \{\infty\}$, if $f(z_0) = z_0$, then z_0 is the a point of f.

A Möbius transformation has at most 3 fixed points.

Definition (fixed point)

Let f be a Möbius transformation, $c \subset \mathbb{C} \cup \{\infty\}$ is a circle, if $f(c) = c$, then c is a fixed circle of f.

Let f, h are Möbius transformations, then $h \circ f \circ h^{-1}$ is conjugate to f. conjugate Möbius transformations map the fixed point of f, z_0 to the fixed point of $h \circ f \circ h^{-1}$, $h(z_0)$. The fixed circle c become $h(c)$.

suppose f has only one fixed point z_0, select $h = \frac{1}{z-z_0}$, then $h \circ f \circ h^{-1}$ is a translation with a single fixed point ∞.

$$\frac{a\infty + b}{c\infty + d} = \infty, c = 0.$$

The fixed circles of $h \circ f \circ h^{-1}$ are parallel lines; those of f are circles tangent at z_0. f is called a parabolic type of transformation.

Möbius Transformation classification

suppose f has two fixed point z_1, z_2, select $h = \frac{z-z_1}{z-z_2}$, then $h \circ f \circ h^{-1}$ maps $\{0,\infty\}$ to $\{0,\infty\}$,

$$\frac{a0+b}{c0+d} = 0, b = 0.$$

$$\frac{a\infty+b}{c\infty+d} = \infty, c = 0.$$

Then $f(z) = wz$, if $w \in \mathbb{R}$, f is a dilation (scaling), hyperbolic type; if $w = e^{i\theta}$, f is a rotation, elliptic type; otherwise, f is called of loxodromic type.

Hyperbolic Möbius Transformation

suppose f is a hyperbolic Möbius transformation. Any circle through two fixed points are fixed circles.

Lemma

Suppose $f: UHP \to UHP$ is a conformal automorphism, then f has two fixed real points.

Proof.

Suppose z is a fixed point, then

$$\frac{az+b}{cz+d} = z, a, b, c, d \in \mathbb{R}$$

therefore $cz^2 + (d-a)z - b = 0$,
$\Delta = (d-a)^2 + 4bc = (a+d)^2 + 1$, it has two real roots z_1, z_2. □

Fixed points and Axis

Define a Möbius transformation $\phi : UHP \to UHP$

$$\phi(z) = \frac{z - z_1}{z - z_2},$$

then ϕ maps z_1, z_2 to $0, \infty$.

$$\phi^{-1} \circ f \circ \phi = sz, s \in \mathbb{R},$$

f maps the y-axis to y-axis.

Definition (Axis of Möbius Transformation)

The circular arc through the two fixed points is called the axis of the Möbius transformation.

Hyperbolic Metric

Definition (Hyperbolic Distance)

Given two points z_1, z_2 in UHP, there exists a unique circular arc γ through z_1 and z_2, and orthogonal to the real axis. Suppose γ intersects the real axis at ζ_1 and ζ_1, ζ_1 is close to z_1, ζ_2 is close to z_2. The hyperbolic distance from z_1 to z_2 is given by

$$d(z_1, z_2) = \log[z_1, z_2; \zeta_1, \zeta_2]^{-1}.$$

Möbius transformations preserve circles, and preserve angles. There exists a unique circular arc through two points and perpendicular to the real axis. Therefore, the above distance is invariant under Möbius transformations.

Theorem

Suppose $f:(M,g)\to(M,g)$ is an isometric automorphism of a Riemannian manifold (M,g), a curve segment γ is in the fixed point set of f, $f(\gamma)=\gamma$, then γ is a geodesic.

Proof.

Choose a point $p\in\gamma$, let X be the unit tangent vector of γ. The $f_*X=X$. Compute the unique geodesic γ_1 through p along X, compute another unique geodesic γ_2 through p and along f_*X. Because f is an isometry, therefore f preserves geodesics. Therefore $f(\gamma_1)=\gamma_2$. Because $X=f_*X$, according the uniqueness of geodesics, $\gamma_1=\gamma_2$. Therefore, γ_1 is in the fixed point set of f, it coincide with γ. □

Geodesic

Corollary

The circular arcs perpendicular to the real axis are hyperbolic geodesics.

Proof.

Let γ be a circular arc, perpendicular to the real axis. γ intersects the real axis at z_0 and z_3. Construct a Möbius transformation

$$\phi(z) = \frac{z - z_0}{z - z_3}$$

Then γ is the fixed points of the Möbius transformation

$$\phi^{-1} \circ f \circ \phi$$

where $f(z) = sz$. □

Hyperbolic Metric

On UHP, hyperbolic metric is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

on the unit disk

$$ds^2 = \frac{dzd\bar{z}}{(1 - z\bar{z})^2}$$

Proof.

from $w = \frac{z - z_0}{1 - \bar{z}_0 z}$, we get

$$dw = \frac{1 - |z_0|^2}{(1 - \bar{z}_0 z)^2} dz,$$

let $w \to z_0$, then we get

$$dw = \frac{dz}{1 - |z_0|^2}$$