Discrete Surfaces

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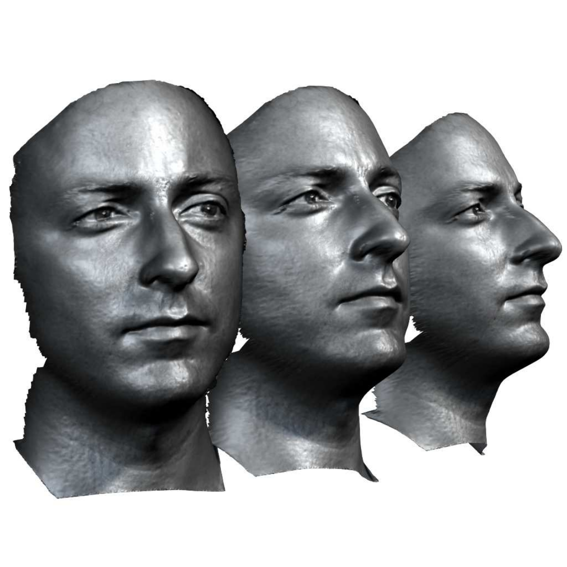
Discrete Surface

Acquired using 3D scanner.

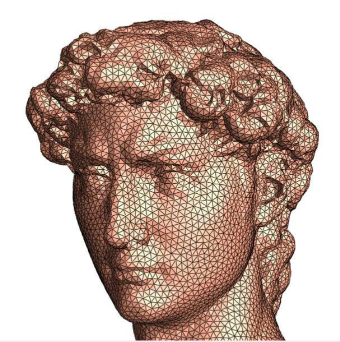


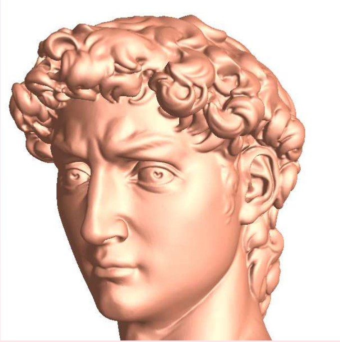
Discrete Surfaces

Our group has developed high speed 3D scanner, which can capture dynamic surfaces 180 frames per second.

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.





Discrete structures

- Topology - Simplicial Complex , combinatorics
- Conformal Structure - Corner angles (and other variant definitions)
- Riemannian metrics - Edge lengths
- Embedding - Vertex coordinates

Simplicial Homology

Definition (Chain Space)

linear combination of simplices $C_k = \{\sum_i \lambda_i \sigma_i^k | \lambda_i \in \mathbb{Z}\}$

Definition (Boundary Operator on a simplex)

$$\partial_n [v_0, v_1, \cdots, v_n] = \sum_k (-1)^k [v_0, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n]$$

Definition (Boundary Operator on a k-chain, $\partial_k : C_k \to C_{k-1}$)

$$\partial_k \sum_i \lambda_i \sigma_i^k = \sum_i \lambda_i \partial_k \sigma_i^k.$$

Definition (Homology Group)

$$H_k = \frac{Ker \partial_k}{Img \partial_{k+1}}$$

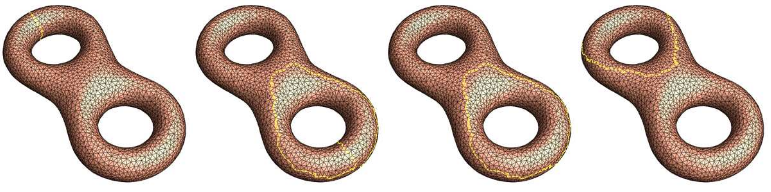
Homology Basis

$H_k(M, \mathbb{Z})$ basis

The eigen vectors corresponding to 0 eigen value of the following linear operator $\Delta_k : C_k \to C_k$:

$$\Delta_k = \partial_k^T \partial_k + \partial_{k+1} \partial_{k+1}^T$$

Homology Group Basis



Simplicial Cohomology

Definition (Cochain)

A k-form ω is a linear functional on C_k , $\omega : C_k \to \mathbb{R}$.

Definition (Cochain Space)

A k-cochain space C^k is the dual space of C_k,
$C^k = \{\omega | \omega k - form\}$

Definition (Discrete Exterior Differentiation)

$d_k : C^k \to C^{k+1}$ linear operator

$$(d_k \omega)(\sigma) = \omega(\partial_k \sigma), \omega \in C^k, \sigma \in C_{k+1}.$$

Definition (Cohomology Group)

$$H^k = \frac{Ker d_k}{Img d_{k-1}}$$

Cohomology Basis

$H^k(M,\mathbb{Z})$ basis

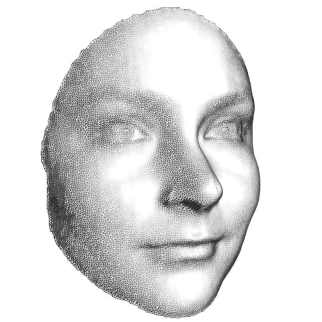
The eigen vectors corresponding to 0 eigen value of the following linear operator $\Delta_k : C_k \to C_k$:

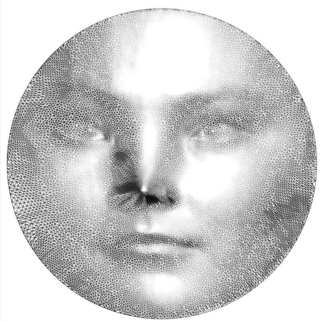
$$\Delta_k = d_k^T d_k + d_{k-1} d_{k-1}^T$$

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite many metrics.





Discrete Curvature

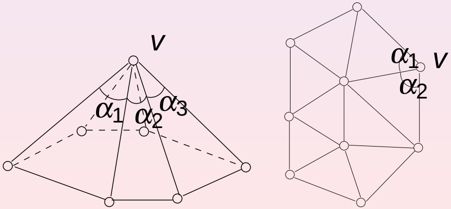
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

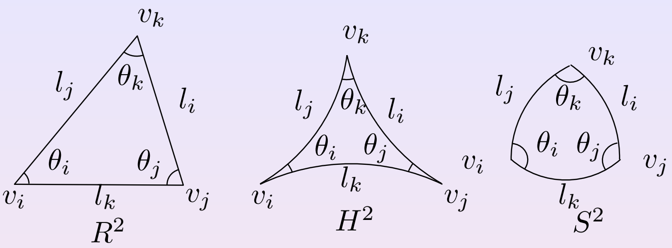
$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Discrete Metrics Determines the Curvatures



Angle and edge length relations: cosine laws $\mathbb{R}^2, \mathbb{H}^2, \mathbb{S}^2$

$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \quad (1)$$

$$\cosh l_i = \frac{\cosh\theta_i + \cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k} \quad (2)$$

$$1 = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \quad (3)$$

Definition (Edge Weight)

Given an edge $[v_i, v_j]$ adjacent to two faces $[v_i, v_j, v_k]$ and $[v_j, v_i, v_l]$, then the edge weight is defined as

$$w_{ij} = \cot\theta_{ij}^k + \cot\theta_{ji}^l.$$

Given a discrete function $f : V \to \mathbb{R}$,

$$\Delta f(v_i) = \sum_j w_{ij}(f(v_j) - f(v_i)).$$

Approximation

Definition (ε-net)

Given a smooth metric surface S, a point set $V \subset S$ is called an ε-net, if $\forall p \in S$, the geodesic disk D_p with radius ε, there must be at least one point $q \in V$.

Definition (Geodesic Delaunay Triangulation)

Suppose S is a metric surface, T is a triangulation, such that all edges are geodesics, and each geodesic circle through the three vertices of each face doesn't contain the fourth vertex.

Theorem

Given a compact metric surface, with an ε-net, if ε is small enough, the geodesic Delaunay triangulation exists.

Closest Point Mapping

Given a compact metric surface S, one can design a random sample set, which is an ε-net, and compute a geodesic Delaunay triangulation. Then, approximate the surface with piecewise Euclidean triangle mesh M. If the original surface is embedded in $\mathbb{R}^3$, then if ε is small enough, then M is contained in a tubular neighborhood of S.

Definition (Closest Point Mapping)

$\forall p \in M$, if ε is small enough, $\exists! q \in S$, such that

$$d(p,q) = \min_{q \in S} d(p,q).$$

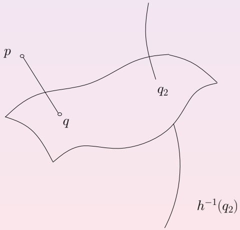
The mapping $h: p \to q$ is called the closest point mappping.

Theorem

If ε is small enough, the closest point mapping is a homeomorphism.

Closest Point Mapping

Suppose p is in the tubular neighborhood of S, $h(p)$ is the closest point on S. Fixing a point q_2 on the surface, then $h^{-1}(q_2)$ is a curve orthogonal to the surface. This gives a parameterization of the tubular neighborhood. If the sample density is high enough, then the mesh is a section of the tubular neighborhood (normal bundle of the surface.)



Theorem (Convergence Analysis)

Let M be the piecewise linear discrete surface induced by an ε-net on a compact metric surface $(S, \mathbf{g}_S)$, the mesh has induced Euclidean metric $\mathbf{g}_M$ if ε is small enough, then $h^{-1}: S \to M$, induces the pull back metric $(h^{-1})^* g_M$ converges to $\mathbf{g}_S$.