Characteristic Class

David Gu[1,2]

[1]Computer Science Department
Stony Brook University
Yau Mathematical Sciences Center
Tsinghua University

Tsinghua University

Characteristic Class

Philosophy

In topology, a geometric or topological being can be easily constructed locally, but when they are generalized to the global, topological obstructions will be encountered. These topological obstructions are usually represented as a cohomology class on the manifold, which are called *characteristic class*.

Example

Suppose S is a closed surface without boundary, genus is not equal to one, then there exists no Euclidean metric, (a Riemannian metric, such that the Gaussian curvature is zero everywhere).

Proof.

According to Gauss-Bonnet,
$\int_S K dA = 2\pi\chi(S) = 2\pi(2-2g)$. □

The characteristic class is the Euler class $\chi(S)$.

Examples

Example

Suppose S is a closed surface without boundary, genus is not equal to one, then there exists no Euclidean atlas, (an atlas, such that all transition functions are rigid motions on the plane).

Proof.

Let (u,v) is an arbitrary local chart, then construct a local metric $g = du^2 + dv^2$. Because S has a Euclidean atlas, then g is globally defined. Contradiction. □

Examples

Example

Suppose S is a closed surface without boundary, genus is not equal to one, then there exists no non-vanishing smooth vector field.

Definition (Index)

Suppose p is an isolated singularity of a vector field. We draw a small loop γ surrounding p, then the mapping $\phi : \gamma \to \mathbb{S}^1$ is given by

$$\gamma(t) \to \frac{v \circ \gamma(t)}{|v \circ \gamma(t)|},$$

then $deg(\phi)$ is called the index of the singularity p.

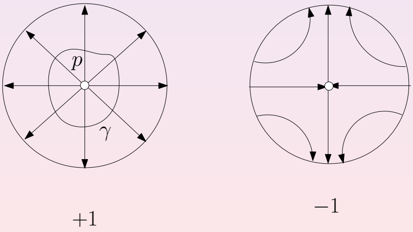


Figure: Singularity Index

Poincaré-Hopf Index Theorem

Theorem (Poincaré-Hopf Index Theorem)

Suppose v is a smooth vector field on a surface S with isolated singularities. The total index

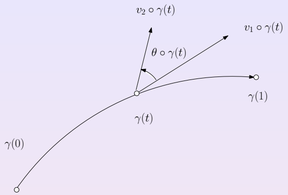
$$\sum_p Ind(p) = \chi(S).$$

Proof.

1. Suppose v_1 and v_2 are two smooth vector fields, then $\sum_{p\in v_1} Ind(p) = \sum_{p\in v_2} Ind(p)$.
2. Construct a special vector field v, such that $\sum_{p\in v} Ind(v) = \chi(S)$.

□

Proof



Proof.

Compute a triangulation, such that each triangle contains at most one singularity either in v_1 or in v_2. Define a 2-form for v_k,

$$\Omega_k([v_0, v_1, v_2]) = Ind(p), p \in [v_0, v_1, v_2],$$

where p is a singularity in $v_k, k = 1, 2$.
Let $\gamma(t)$ be a curve segment. The angle $\theta \circ \gamma(t)$ is the angle from $v_1 \circ \gamma(t)$ to $v_2 \circ \gamma(t)$. □

Proof

Proof.

Each edge is represented as a curve $\gamma : [0,1] \to \mathbb{R}$. Define one form

$$\omega = \int_0^1 \frac{d\theta}{ds} ds.$$

Then

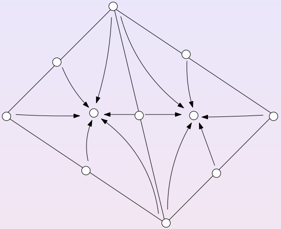
$$\Omega_2 - \Omega_1 = d\omega.$$

The total index is given by

$$\int_S (\Omega_2 - \Omega_1) = \int_S d\omega = \int_{\partial S} \omega = 0.$$

□

Special Vector Field



Proof.

Construct a canonical vector field based on a triangulation. Each vertex is a singularity with index $+1$, each face is also a singularity with index $+1$, each edge is a singularity with index -1. The total index is $\chi(S)$. □

Unit Tangent Bundle

Consider all the unit tangent vectors of a topological sphere $\mathbb{S}^2$. Use Stereo-graphic projection, we can parameterize the sphere without the north pole. Each point in the unit tangent bundle is represented as (z, dz).
We do stereo-graphic projection from the south pole, to get another chart (w, dw). The coordinate transition function is given by

$$w = \frac{1}{z}, dw = \frac{-1}{z^2}dz.$$

The unit tangent bundle for each hemisphere is a direct product

$$\mathbb{D}^2 \times \mathbb{S}^1,$$

which is a solid torus.
We need to glue the two solid tori along their boundaries, $f: T^2 \to T^2$. Each fiber is glued with a fiber, but the two fibers differ by a rotation angle. Select a point on the equator $p = e^{i\theta}$, then $dw \to e^{i(2\theta+\pi)}dz$.

$$f(a) = a, f(b) = b - 2a.$$

Unit Tangent Bundle

Suppose $[\gamma] \in H_1(T_1, \mathbb{Z})$ is the b generator, $[f(\gamma)] \in H_1(T_2, \mathbb{Z})$ is $b - 2a$.
A smooth vector field without singularity is a smooth surface S in the UTM, such that S intersect each fiber at one point. Such a surface is called a global section.
Intuitively, in the solid torus γ can shrink to a point, $f(\gamma)$ is not homologous to 0, so it can not bound a surface. The global section doesn't exist.

Topological obstruction

1. Compute a triangulation of the initial surface Σ, such that each triangle is small enough, the restriction of the unit tangent bundle on the triangle is trivial (direct project).
2. For each vertex v, choose a point $s(v)$ in its fiber $p^{-1}(b)$.
3. For each edge $[v_i, v_j]$, in the trivial neighborhood, $[v_i, v_j] \times \mathbb{S}^1$ interpolate $s(v_i)$ and $s(v_j)$.
4. For each face $f := [v_i, v_j, v_k]$, in the trivial neighborhood, $[v_i, v_j, v_k] \times \mathbb{S}^1$, compute the degree of the map
$$\phi_f : s(\partial [v_i, v_j, v_k]) \to \mathbb{S}^1,$$
if degree is zero, then the section can be extended to the interior of the face, otherwise, we encounter an obstruction.
5. The two form
$$\Omega(f) = deg(\phi_f),$$
is the topological obstruction class.

Fixed Point

Lemma (Brower Fixed Point)

Suppose $f : \mathbb{D} \to \mathbb{D}$ is a continuous map, which maps the boundary of the disk to the boundary of the disk, then there exists a fixed point.

Proof.

Assume there is no fixed point, then draw a ray starting from $f(p)$ through p and intersects the boundary at p, this gives a continuous map $\phi : p \to q$, which maps the whole disk to its boundary, the restriction of ϕ on the boundary is the identity. Contradiction. □

Fixed Point

Suppose $f : S \to S$ is a continuous map homotopic to the identity, which maps S to itself. We can use a simplicial map to approximate it. Therefore, we assume S is a simplicial complex, f is a simplical map. Furthermore, we can assume f has isolated fixed points. $f_k : C_k \to C_k$ is a linear map, and can be represented as matrices.
According to Brower's fixed point lemma,if $f_0(v_i) = v_i$ then v_i is a fixed point; if $f_1([v_1, v_2]) = [v_1, v_2]$, then there exists a fixed point in the edge $[v_1, v_2]$; if $f_2([v_1, v_2, v_3]) = [v_1, v_2, v_3]$, then there exists a fixed point inside the face. But fixed points are over counted. Therefore, the total number of fixed points is given by

$$tr(f_0) - tr(f_1) + tr(f_2).$$

Because f is homotopic to identity, we can use identity for the computation, therefore the above is

$$dim(C_0) - dim(C_1) + dim(C_2) = \chi(S).$$

Fixed Point

Consider the sequence

$$C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

therefore

$$dim(C_k) = dim(ker\,\partial_k) + dim(img\,\partial_k)$$

So the dimension satisfies the following:

$$\begin{aligned}\chi(S) &= ker\,\partial_2 + Img\,\partial_2 - ker\,\partial_1 - img\,\partial_1 + ker\,\partial_0 + 0\\ &= ker\,\partial_2 - (ker\,\partial_1 - img\,\partial_2) + (ker\,\partial_0 - img\,\partial_1)\\ &= H_2(S,\mathbb{Z}) - H_1(S,\mathbb{Z}) + H_0(S,\mathbb{Z})\end{aligned}$$

fixed point

All the tangent vectors form the tangent bundle TM of the surface. Each point is represented as $(p, v(p))$, where $v(p) \in TM_p$. The 0 section is $(p, 0)$, $(p, \varepsilon v(p))$ is a perturbation of the 0-section. All the vector fields has $\chi(S)$ zero points. Therefore, the 0-section has algebraic $\chi(S)$ self-intersections in the tangent bundle.

fixed point

Let S be a surface, then the neighborhood of the diagonal (p,p) is homeomorphic to the tangent bundle TM. The diagonal (p,p) corresponds to the 0-section. The self-intersection number of the diagonal is the Euler number.
Note that, given a triangle mesh M, then the direct product $M \times M$ can be easily constructed, the boundary operator, the homology, cohomology can be easily computed.

Isotopy

Given two knots embedded in $\mathbb{R}^3$, verify if one can deform to the other in $\mathbb{R}^3$.

Given two surfaces embedded in $\mathbb{R}^3$, verify if one can deform to the other in $\mathbb{R}^3$ without self-intersection.

Isotopy

$f_0 : S \hookrightarrow \mathbb{R}^3$, then consider the following mapping $(f_0, f_0) : S \times S \hookrightarrow \mathbb{R}^3 \times \mathbb{R}^3$, we use F_0 to denote (f_0, f_0), then the preimage of the diagonal is the diagonal. We use Δ_S to denote the diagonal of $S \times S$, $\Delta_{\mathbb{R}^3}$ the diagonal of $\mathbb{R}^3 \times \mathbb{R}^3$. Then

$$F_k : S \times S - \Delta_S \hookrightarrow \mathbb{R}^3 \times \mathbb{R}^3 - \Delta_{\mathbb{R}^3},$$

Suppose $[M]$ is the generator of $H^2(\mathbb{R}^3 \times \mathbb{R}^3 - \Delta_{\mathbb{R}^3}, \mathbb{R})$, if f_0 and f_1 are isotopic, then

$$F_0^*[M] = F_1^*[M].$$

This is called the characteristic class of isotopy.