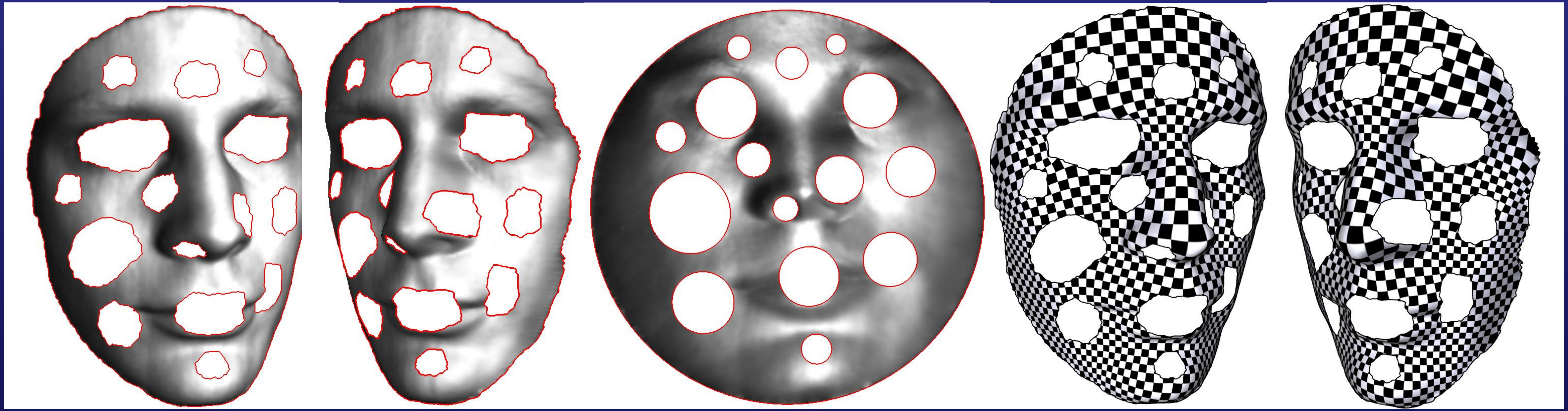
Generalized Koebe's Method for Conformal Mapping Multiply Connected Domains

Wei Zeng [1] Xiaotian Yin [1] Min Zhang [1] Feng Luo [1] Xianfeng Gu [1]

[1] *State University of New York at Stony Brook* [2] *Rutgers University*

Problem

- *Circular conformal mapping for Multiply connected domains*



Highlights

- A practical algorithm to explicitly construct *conformal mappings* for *multiply connected domains*.
- *Koebe's Uniformization Theory*

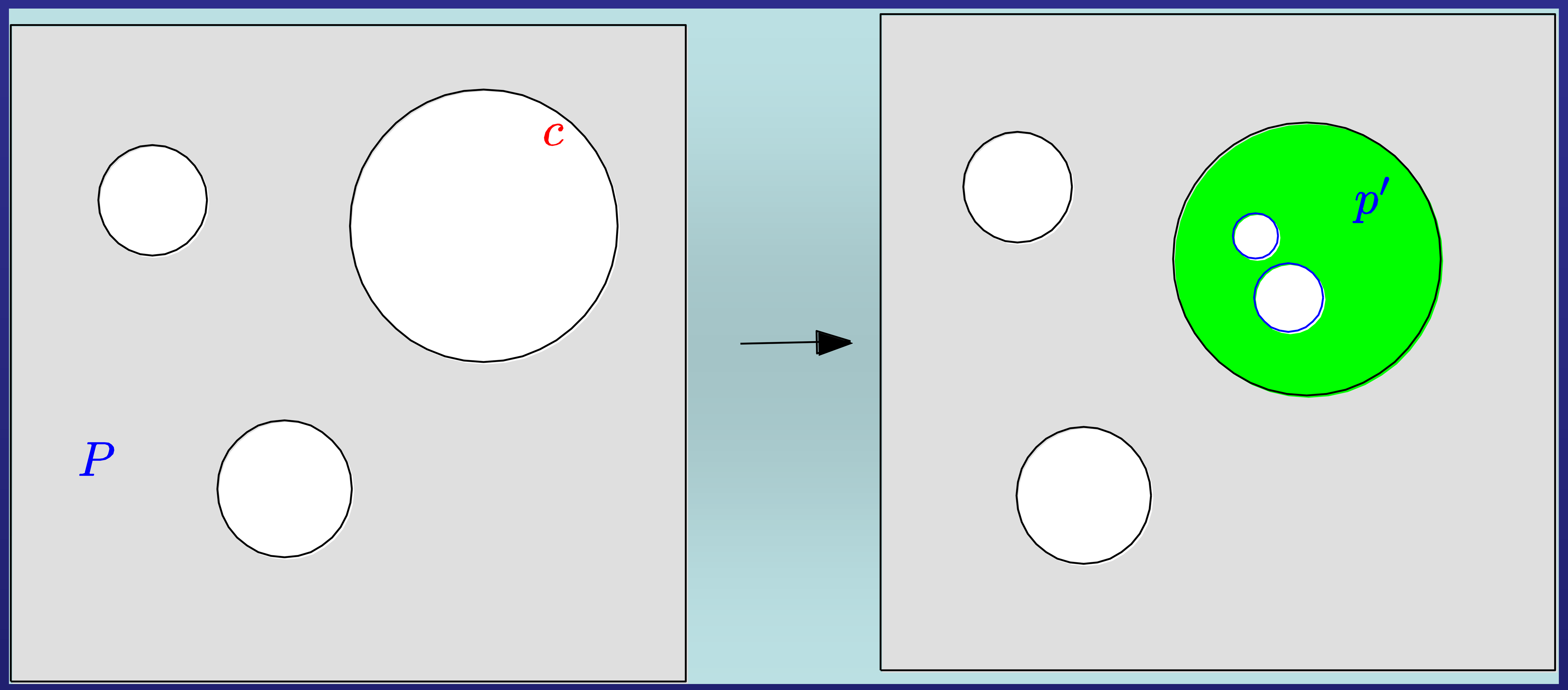
  "All genus zero multiply connected surfaces can be mapped to a planar disk with multiply circular holes. This kind of mappings are angle preserving and differ by Mobius transformation."
- *Generalized Koebe's Method*

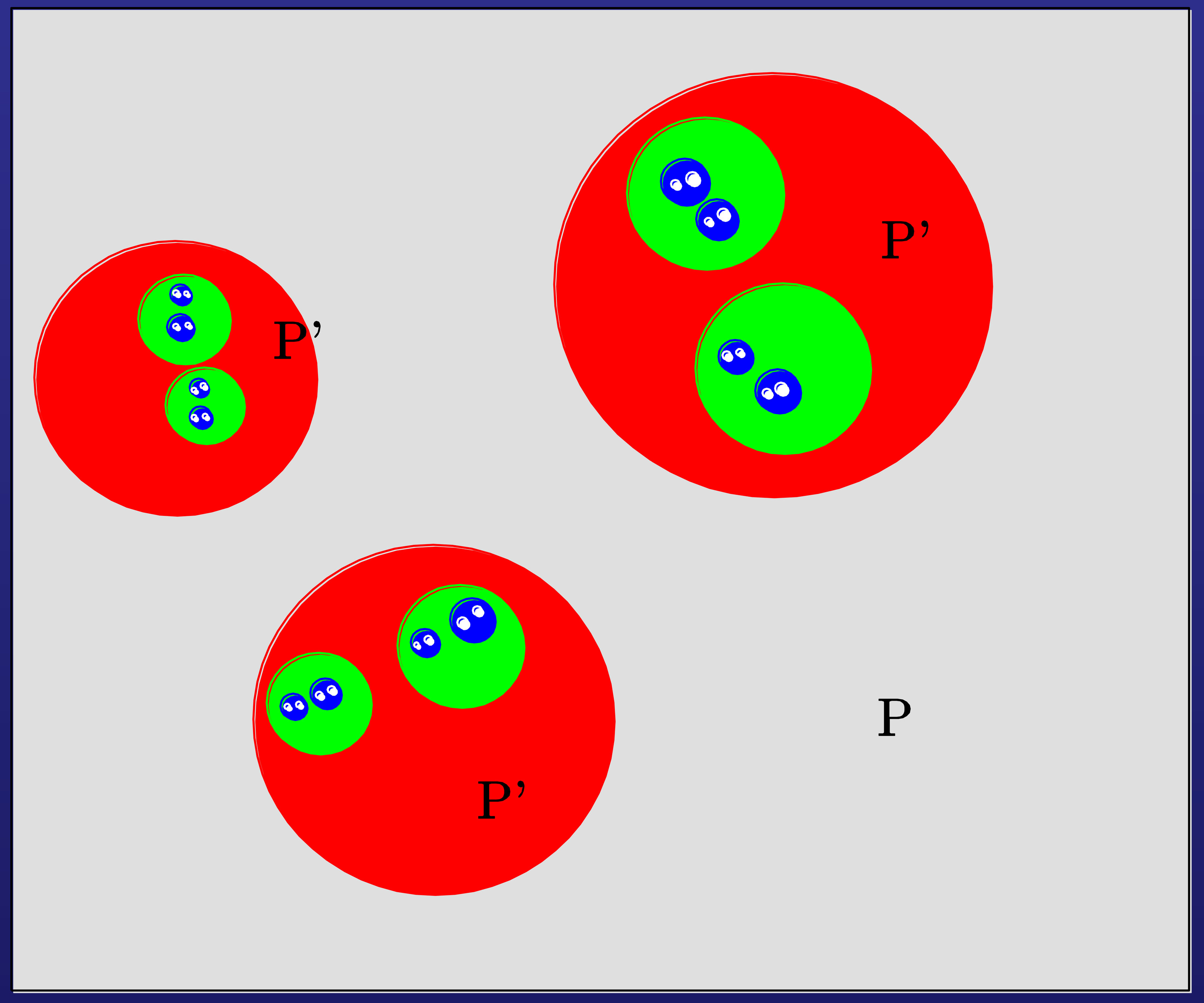
Advantages

- General for *multiply connected* *Domains/Surfaces*.
- Efficient from *iterative construction of* *linear* *steps*.
- Rigorous *proof for the* *exponential* *converging rate*.
- Intrinsic to *conformal structure of surface*.
- Free of *angle distortion*.

Koebe's Theorem

- Schwartz reflection & Schottky group





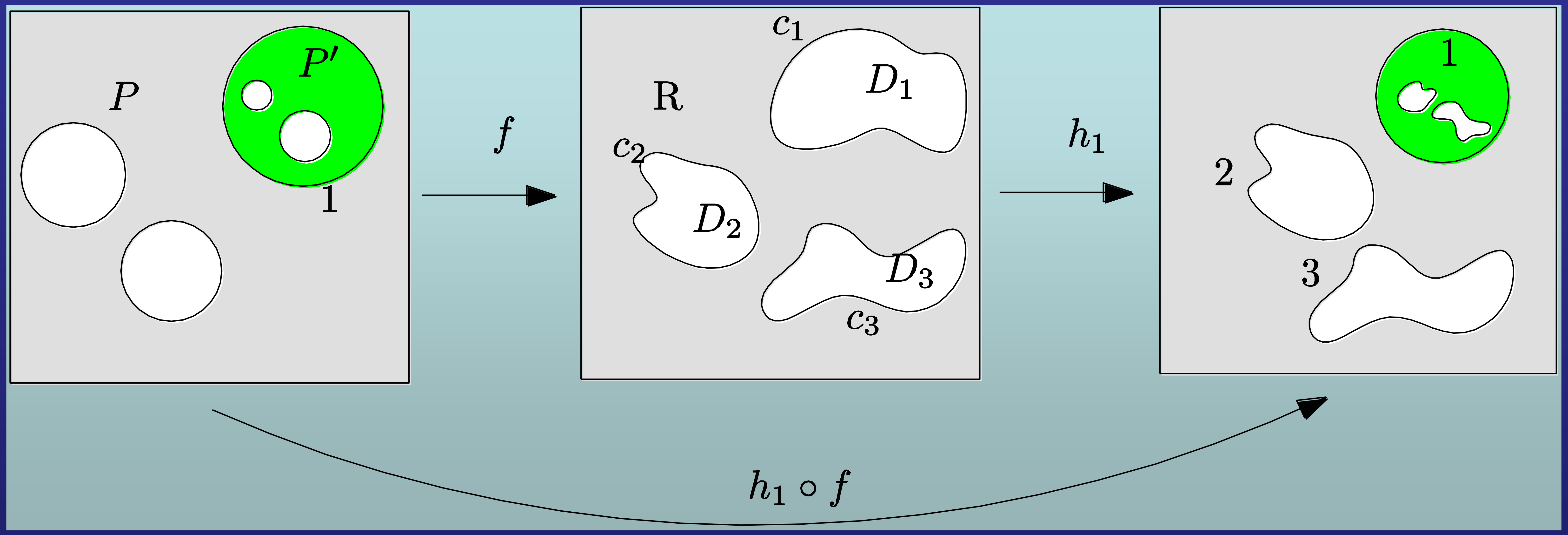
Theorem [Henrici 1993]

Suppose the planar surface has n boundaries, then there exist constants $C_1>0$, $0<C_2<1$, for step k, for all z in P,

$$\phi(z) = z + O\left(\frac{1}{z}\right), z \to \infty, \qquad |\phi(z) - z| < C_1 C_2^{[\frac{k}{n}]}$$

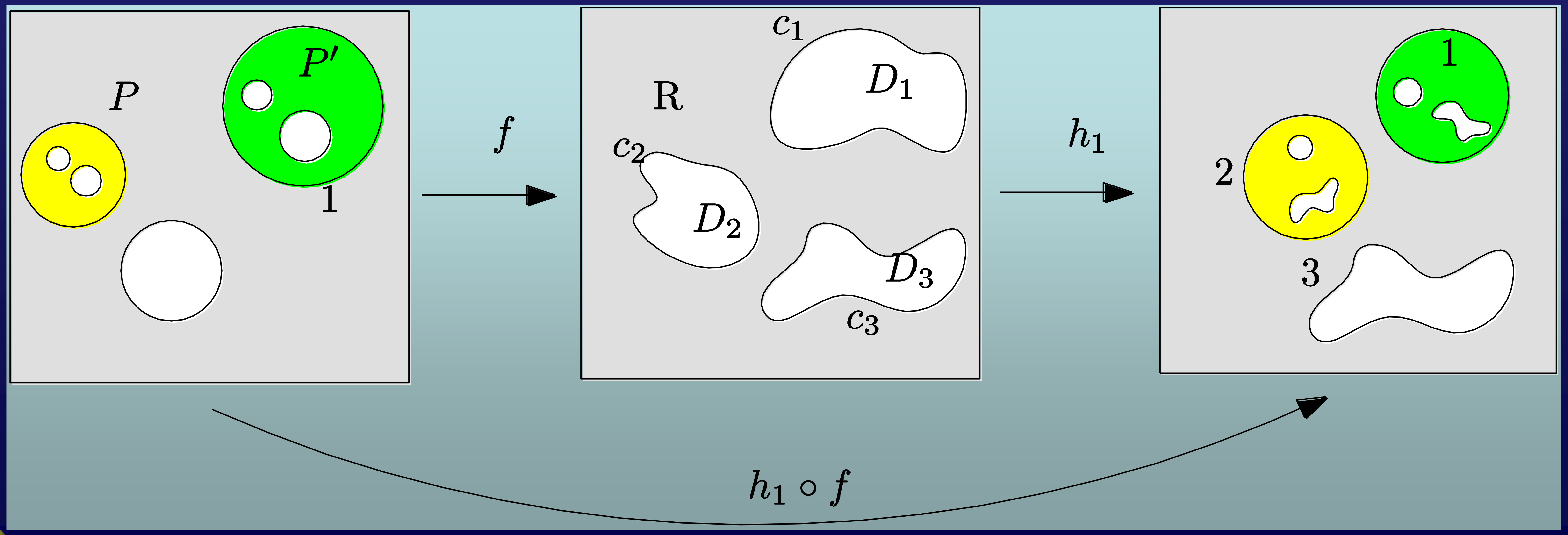
Koebe's iteration and convergence

- Conventional Koebe's method



$$|h_k \circ f(z) - z| < C_1 C_2^{[\frac{k}{n}]}$$

- Generalized Koebe's method



$$|h_k \circ f(z) - z| < C_1 C_2^{2[\frac{k}{n}]}$$

Previous work

- Surface parameterization
  - angle / area distortion
- Conformal parameterization
  - angle preserving
  - desirable for engineering applications
- Discrete harmonic map [Pinkall & Polthier 1993]
  - convex target domain, homeomorphism
- LSCM [Levy et al. 2002]
  - 2 specified features, free boundary

Previous work (cont.)

- Discrete Ricci flow method [Jin et.al 2008]
  - – the only existing one for multiply connected domains
  - – Highly non-linear

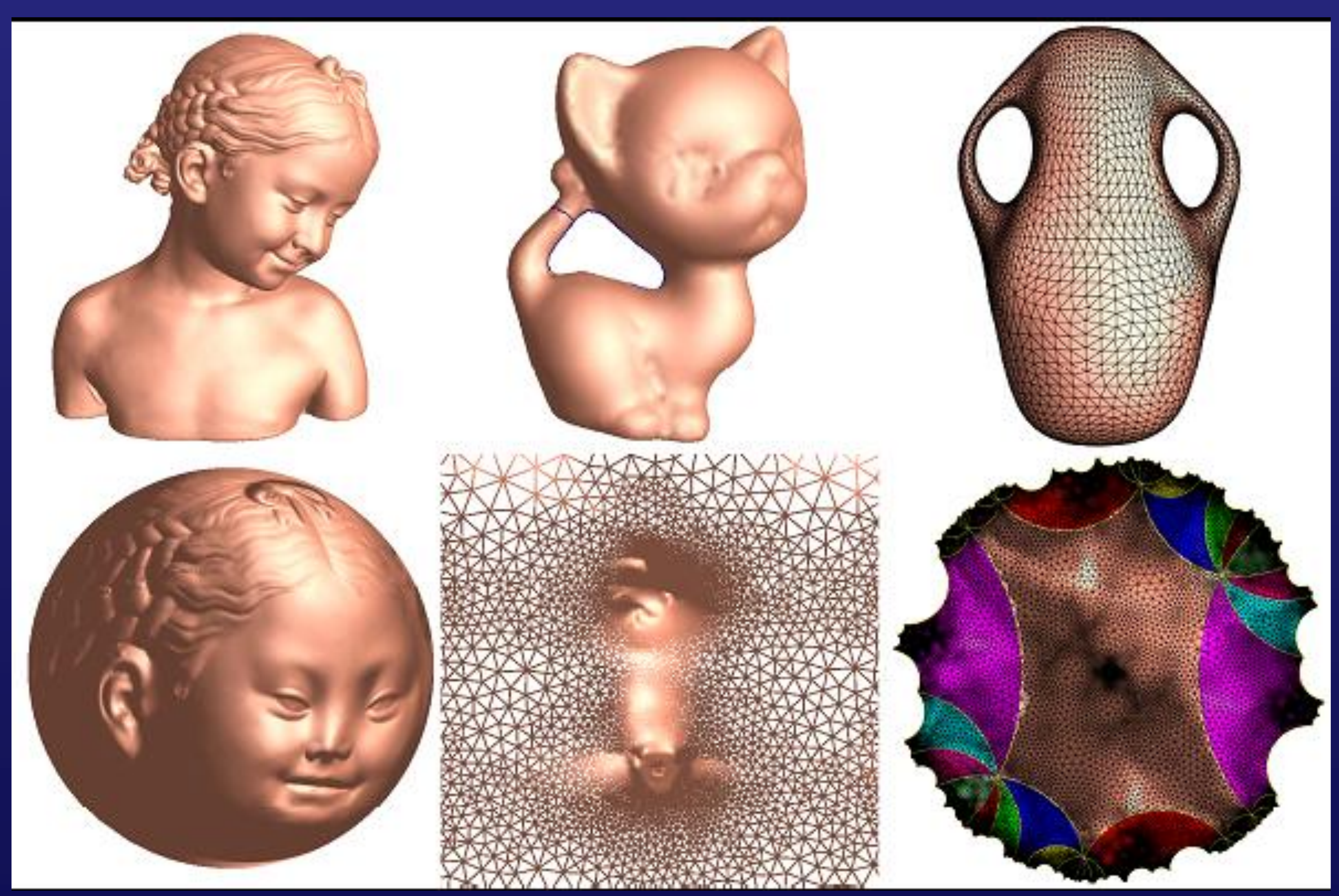
3-holed **Harmonic Map** **LSCM** **Ricci Flow**

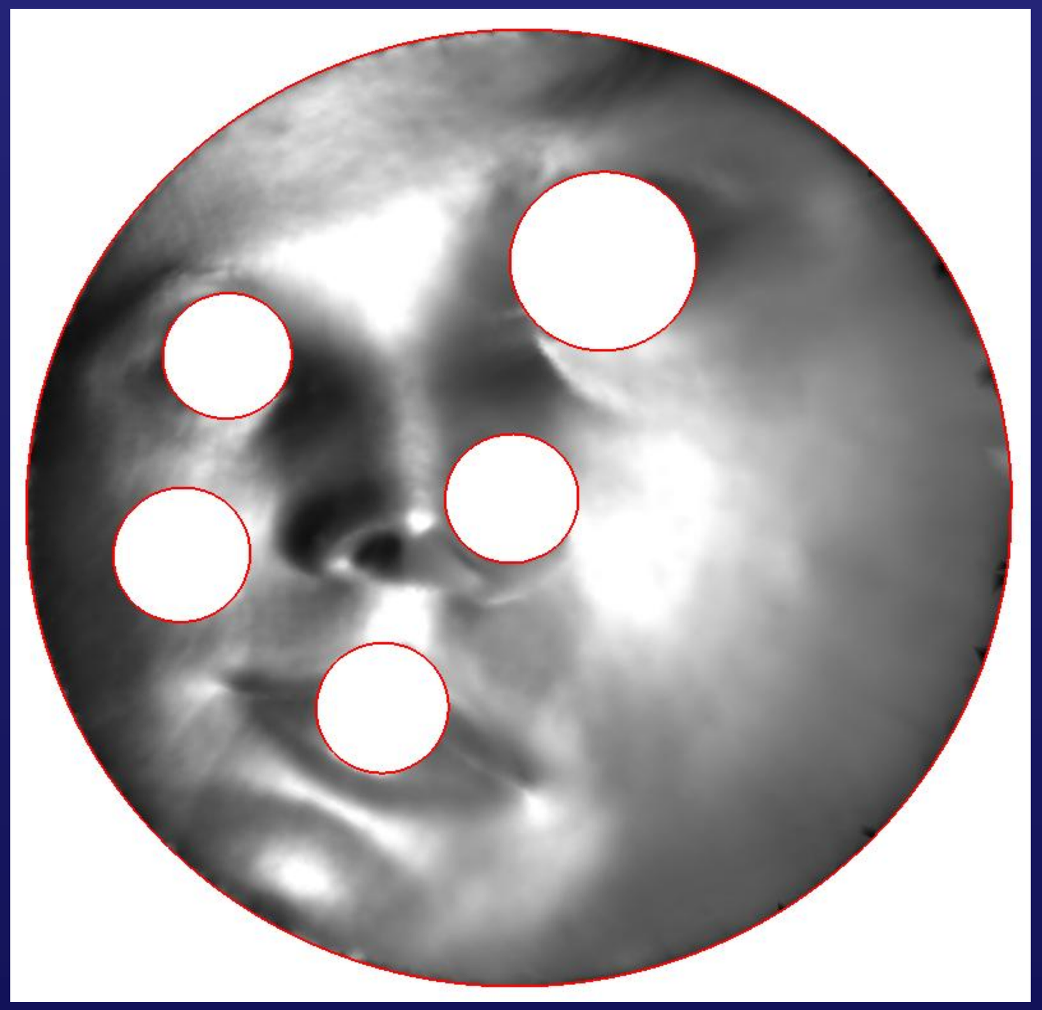
Previous work (cont.)

- Discrete holomorphic 1-form method
  - holomorphic 1-form [Gu & Yau 2003]
  - discrete exterior calculus [Mercat 2004]
  - generalized 1-forms [Gortler et al. 2005]
  - ….
  - linear
  - *Applications: surface tiling, quad remeshing, surface matching, manifold splines…*

Conformal uniformization

- Conformally map any domains / surfaces to canonical shapes.
- Riemann uniformization theorem.





Doubly connected domain

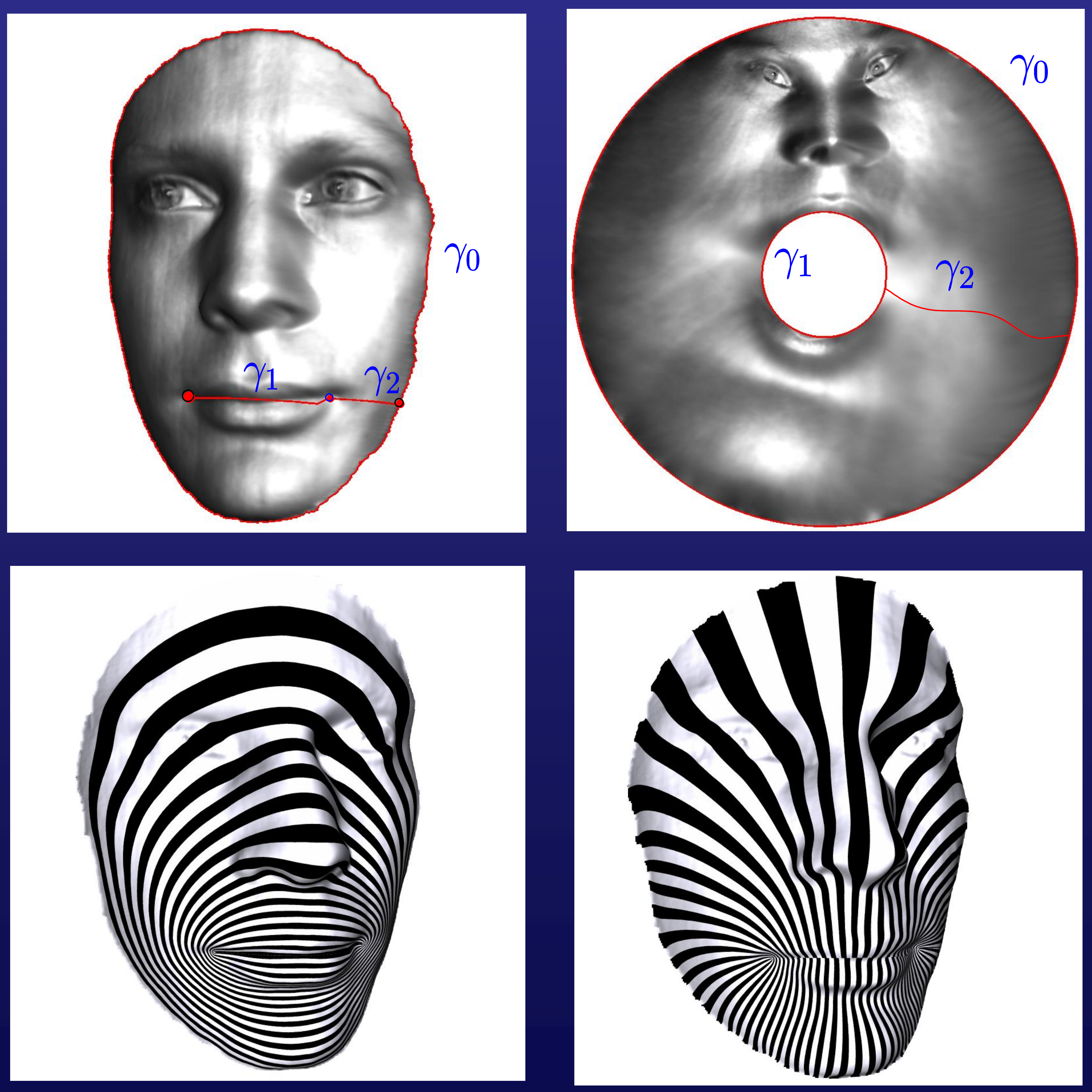
- *Harmonic 1-form* df
- *Conjugate harmonic 1-form*
  $*df = \lambda(dg_0 + dg_1)$

$$f : S \to R$$
$$\begin{cases} f_{\gamma_0} = 0 \\ f_{\gamma_1} = 1 \\ \Delta f = 0 \end{cases}$$

$$g_0 : \tilde{S} \to R$$
$$\begin{cases} g_0 |_{\gamma_2^+} = 1 \\ g_0 |_{\gamma_2^-} = 0 \end{cases}$$
$$\delta(dg_0 + dg_1) = 0$$

- *Holomorphic 1-form*
  $\omega = df + i\, *df$
  $\quad = df + i\lambda(dg_0 + dg_1)$

Annulus Mapping

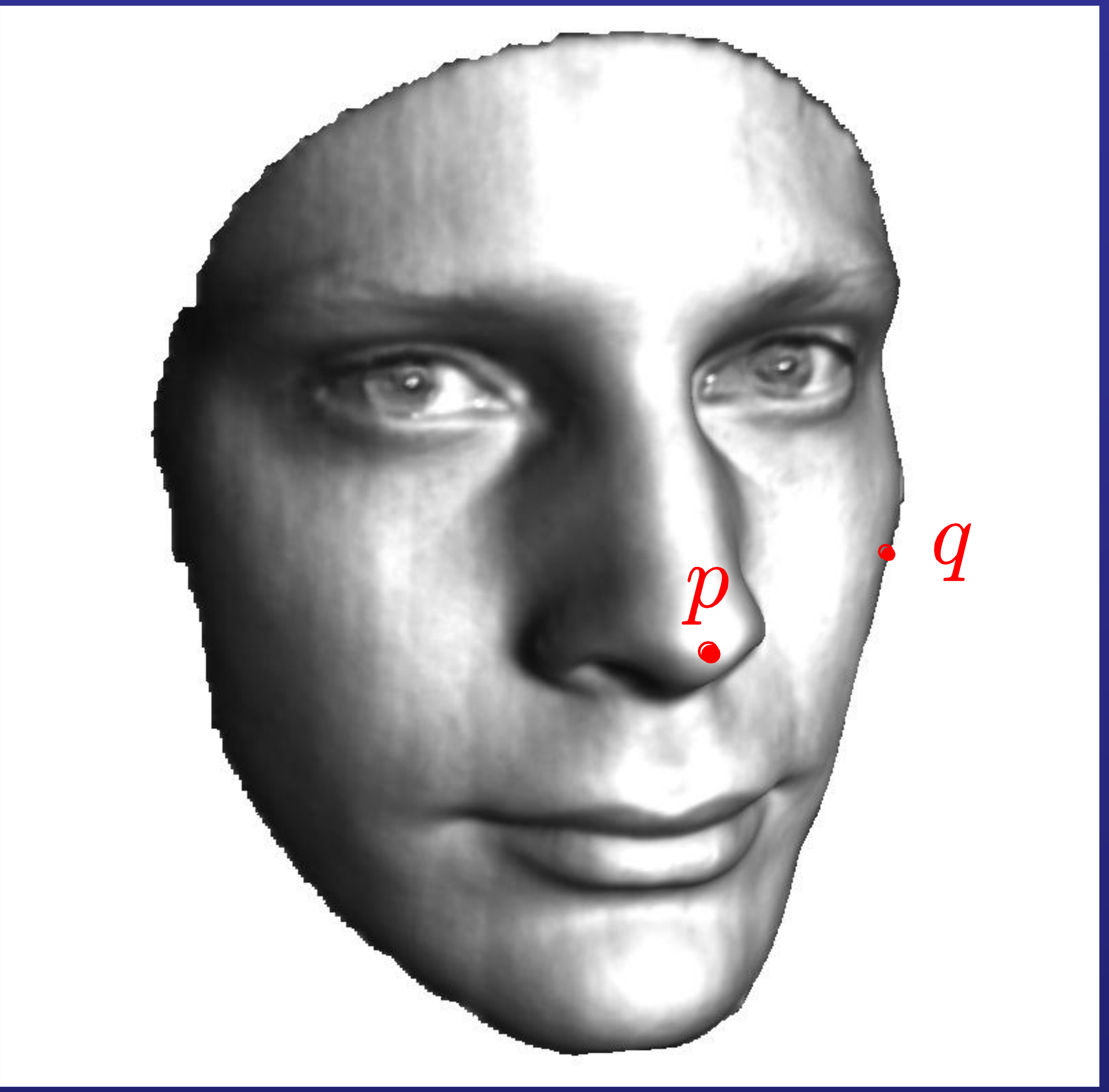


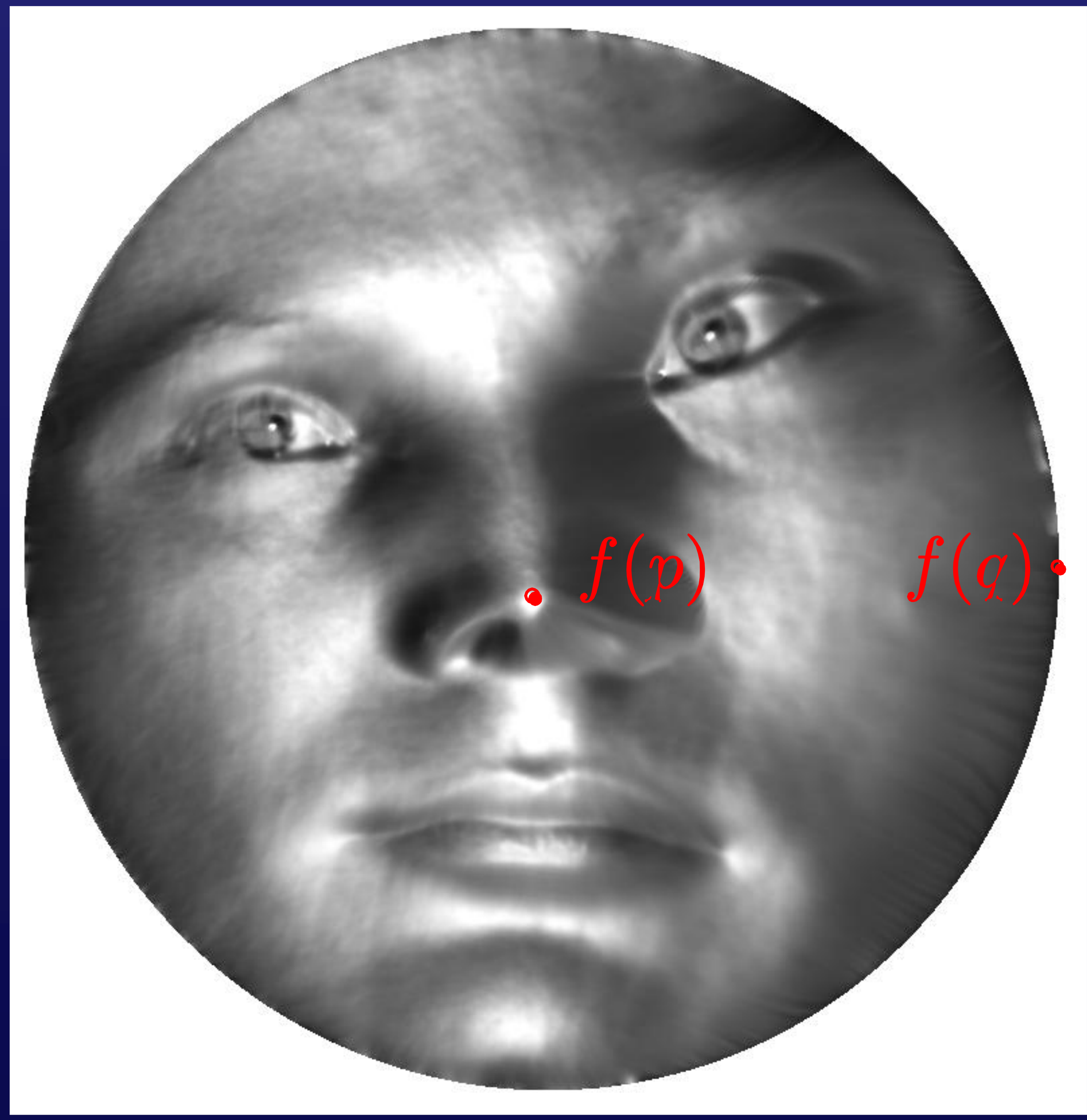
Simply connected domain

- *Riemann Mapping*
  - an interior point p
  - a boundary point q
  - A sequences of small disks D_n
  - Conformal mappings

$$f_n : S - D_n \to D$$

Disk Mapping





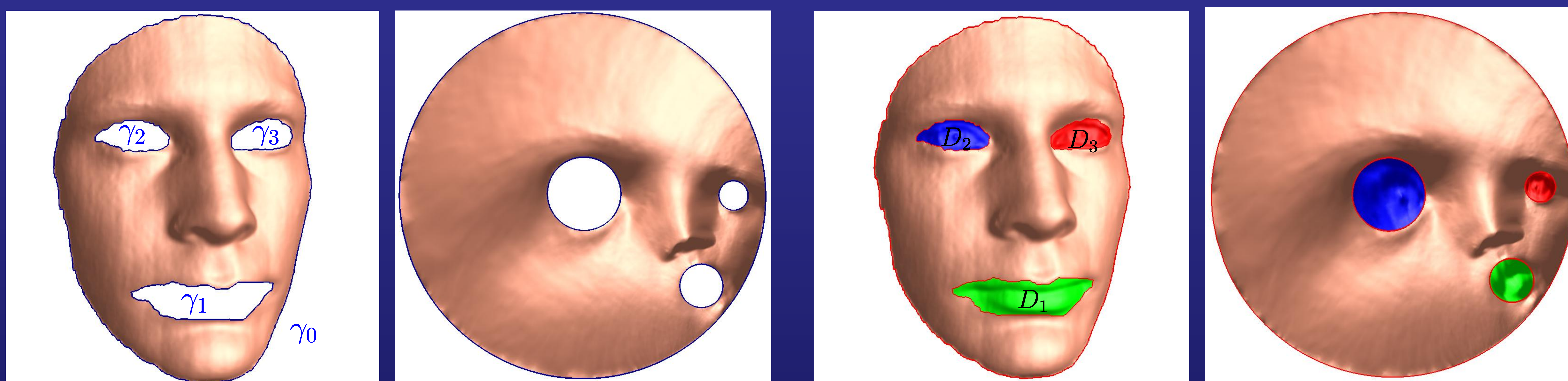
Theorem: *The mappings $\{f_n\}$ converge to the Riemann mapping*

$$f : S \to D, \ \lim_{n\to\infty} f_n = f$$

such that f maps p to the origin, q to 1.

Multiply connected domain

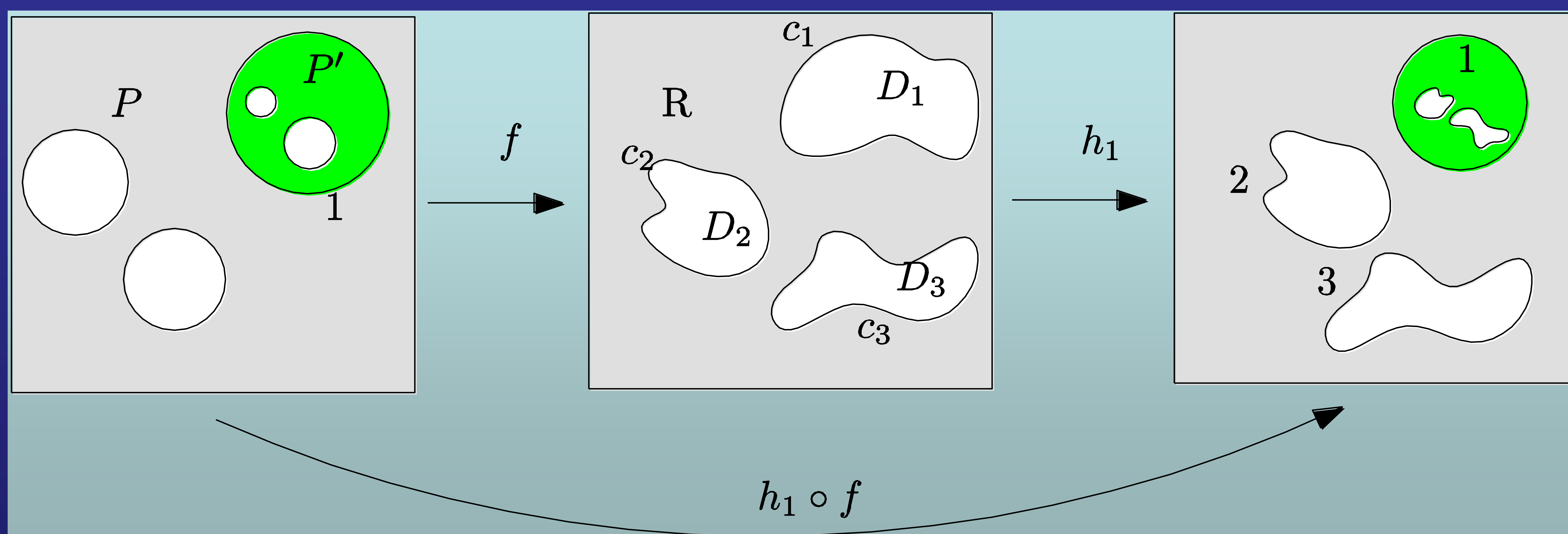
- *n*-holed genus 0 surfaces (*n*>1)



- *Previous*: Ricci curvature flow methods
- *Proposed*: Holomorphic 1-form based Koebe's methods

Proof of Koebe's methods

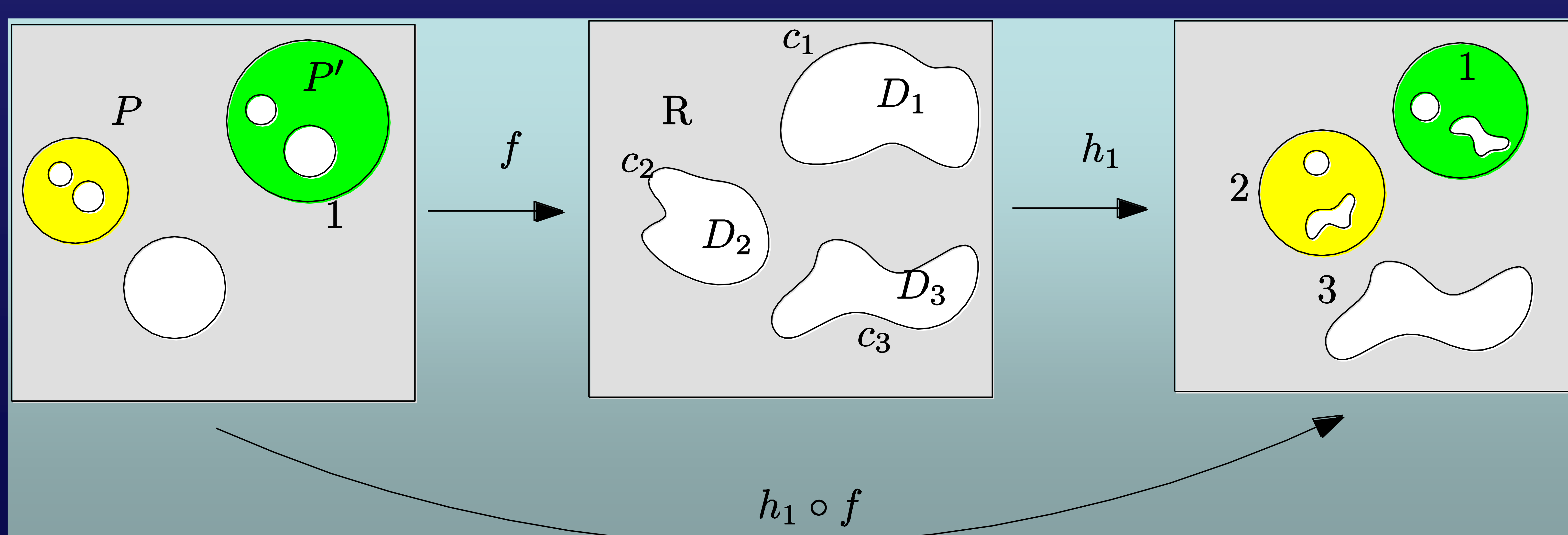
- Conventional Koebe's method



Disk Mapping

$$|h_k \circ f(z) - z| < C_1 C_2^{[\frac{k}{n}]}$$

- Generalized Koebe's method



Annulus Mapping

$$|h_k \circ f(z) - z| < C_1 C_2^{2[\frac{k}{n}]}$$

Conventional Koebe's method (CK)

Multiply connected domain *S*, *n* boundaries

- Composed by *linear iterations*
- Each iteration includes *n steps*
- Each step maps *1 boundary* to the unit circle.
- Each step has *linear* time, *angle preserved*.

Theorem proof (Henrici)

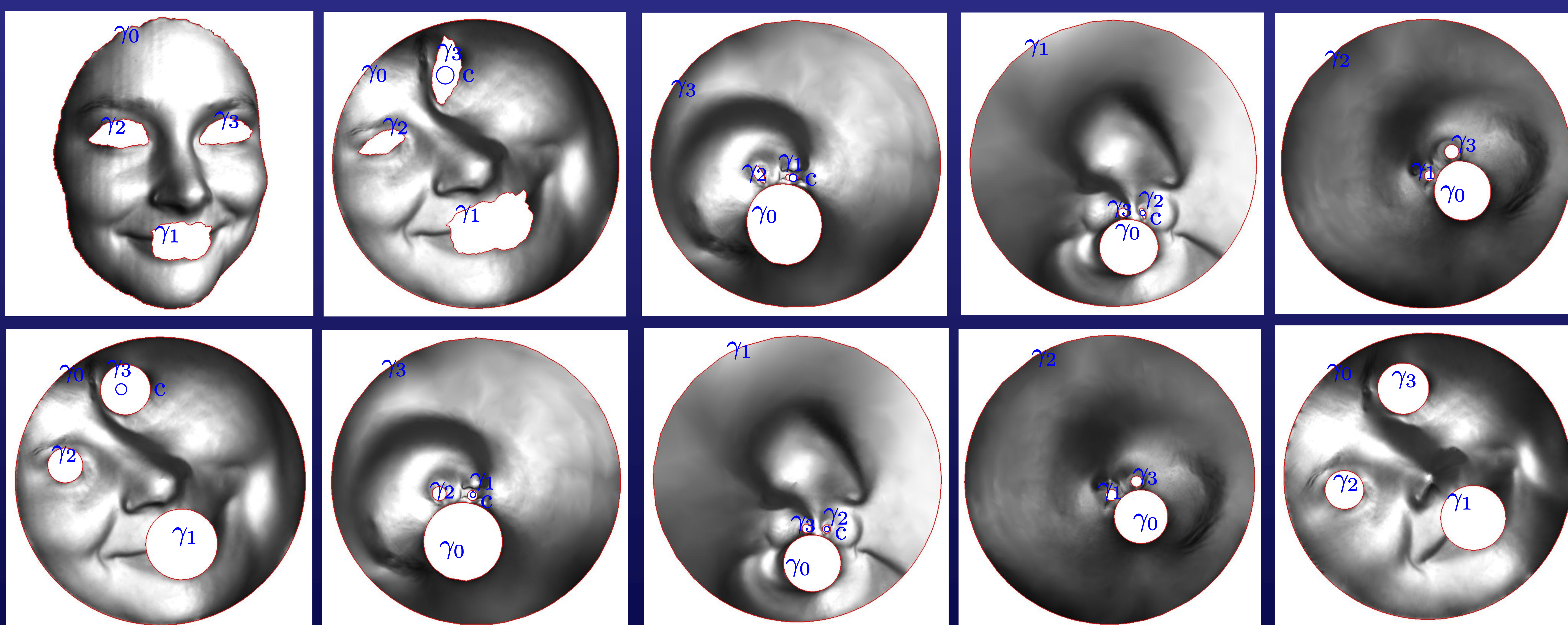
- Convergence is proved using Cauchy intergral formula and area estimate.
- See pages 502-505 in [Henrici 1993]

CK - step

- Conformally map 1 inner boundary γ_3 to the unit circle *D*.
- Linear time

CK - iteration

- In each iteration, each boundary is chosen to be mapped to the unit circle. $(\gamma_0, \gamma_3, \gamma_1, \gamma_2)$



2 iterations

Generalized Koebe's method (GK)

Multiply connected domain *S*, *n* boundaries

- Composed by *iterations*
- Each iteration includes *n/2 steps*
- Each step maps *2 boundaries* to circles.
- Each step has *linear* time, *angle preserved*.

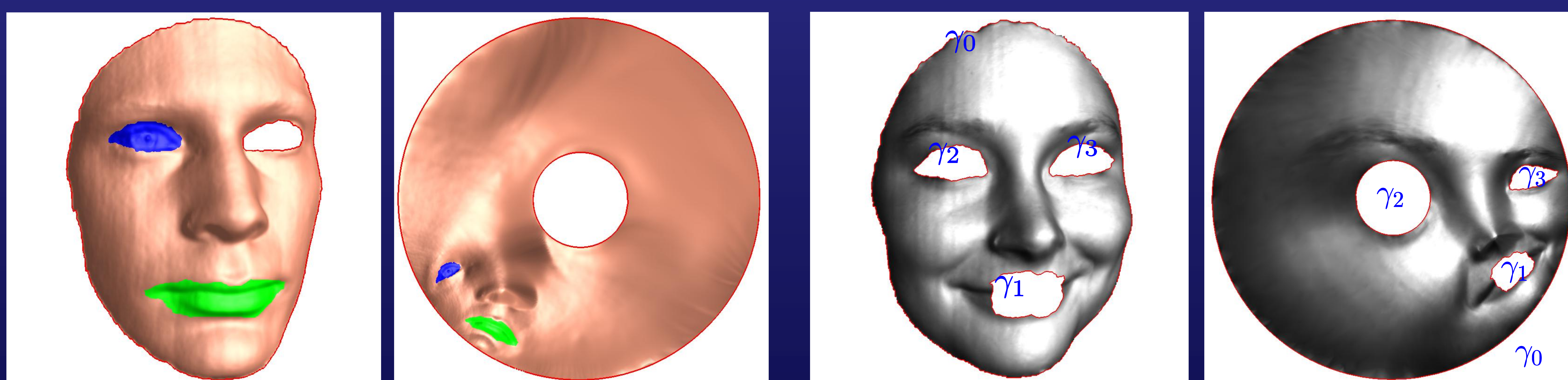
Much faster &
Much fewer **!!**

Theorem proof

- Convergence is proved using Cauchy intergral formula and area estimate by reflection.
- Appendix

GK – step

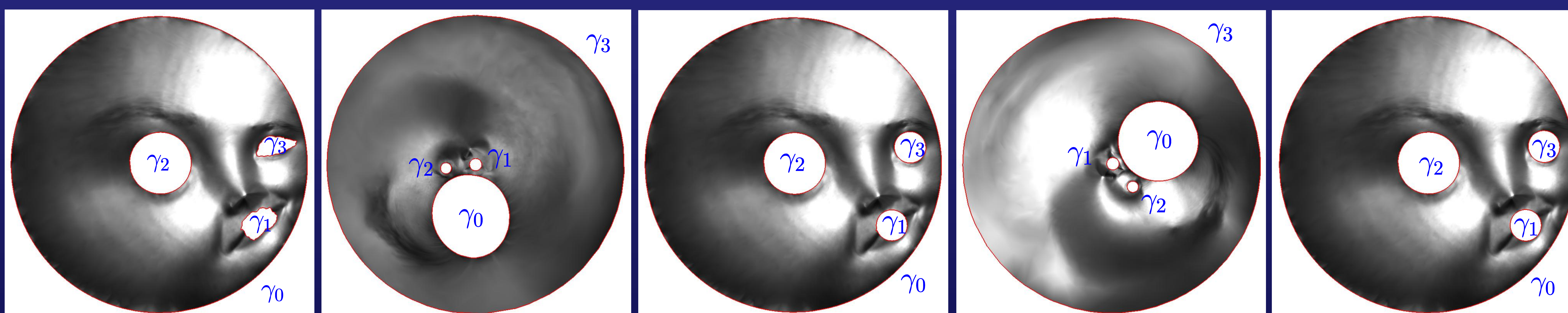
- Conformally map 2 boundaries γ_0, γ_2 to the interior and exterior circles of unite disk *D*.
- Linear time



GK - iteration

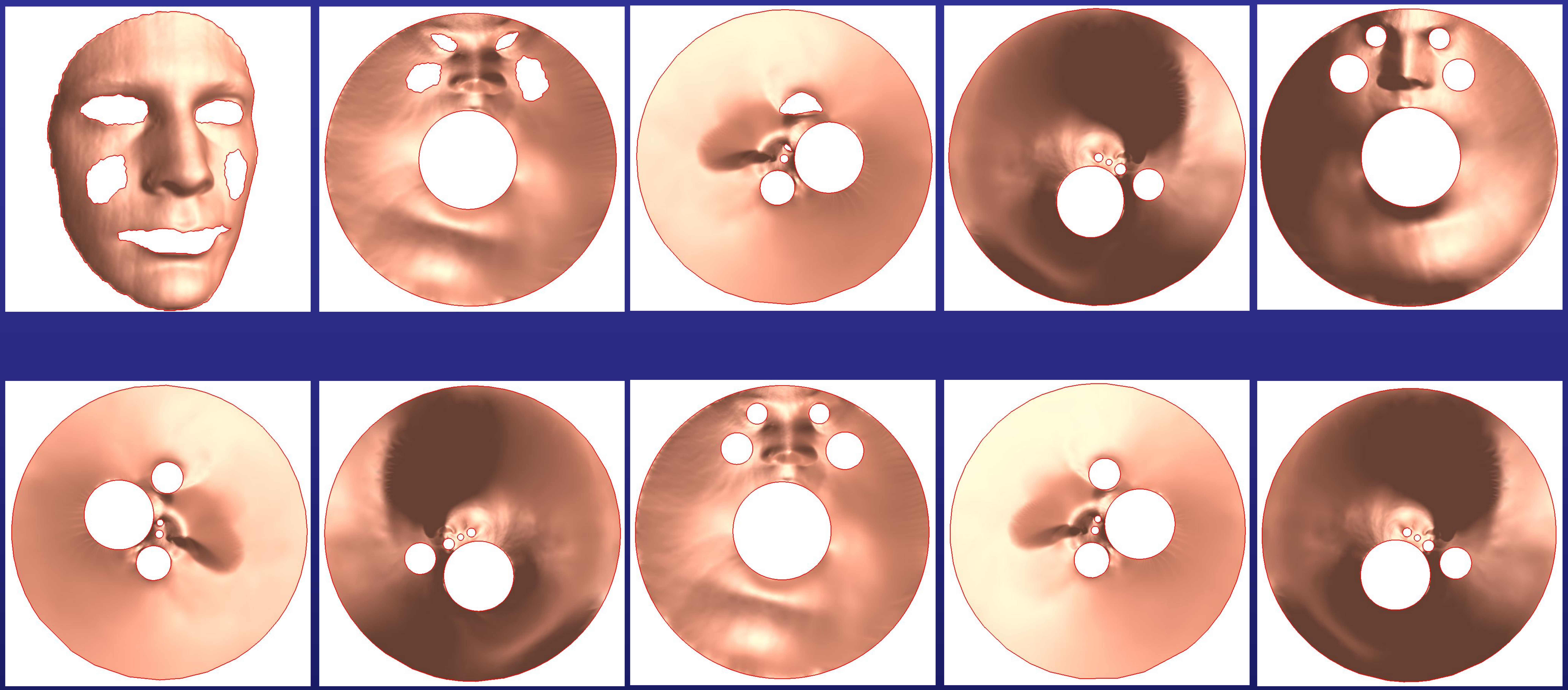
- In each iteration, each two boundaries are chosen to be mapped to the exterior and interior circles.

(γ_0, γ_2) (γ_3, γ_1) (γ_0, γ_2) (γ_3, γ_1) (γ_0, γ_2)



2 iterations

Each step chooses 1 inner boundary mapped to inner circle.
The outer boundary is always mapped to exterior unit circle.



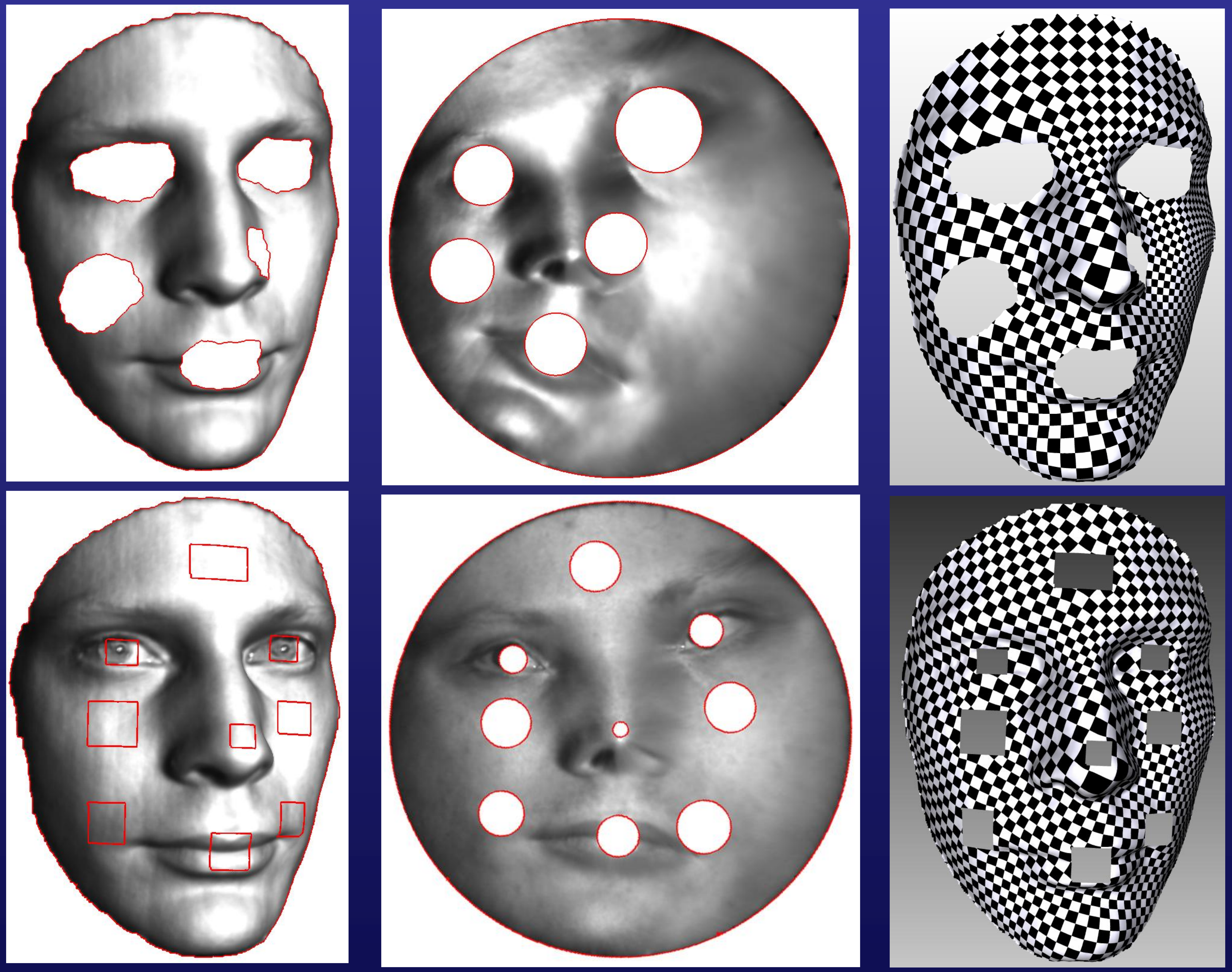
Each step chooses arbitrary 2 boundaries mapped to exterior and interior circles.

The outer boundary is regarded as a hole and filled during the computation.

3 iterations

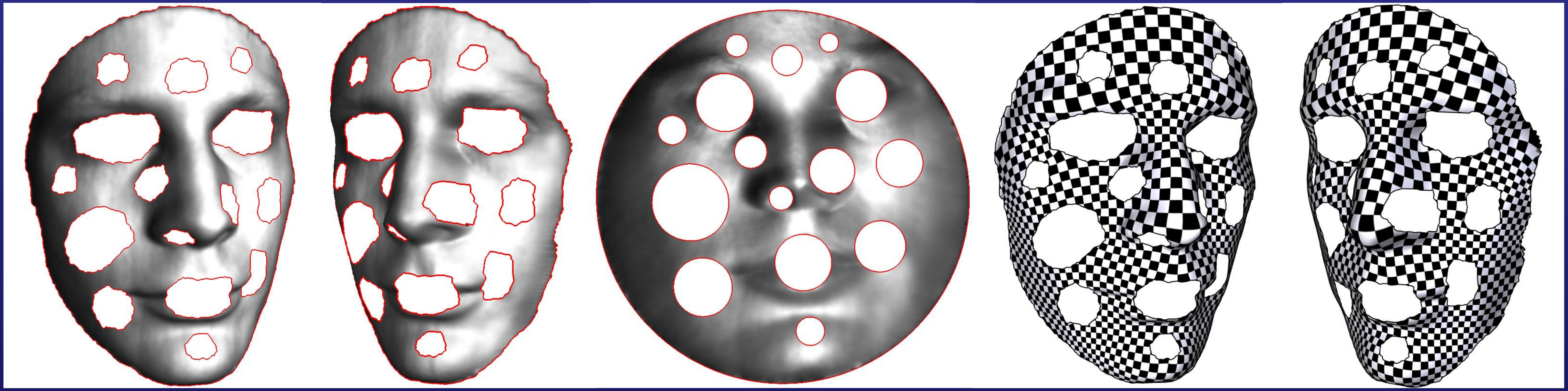
Conformal Maps of Koebe's M.

- 6 boundaries
- 10 boundaries



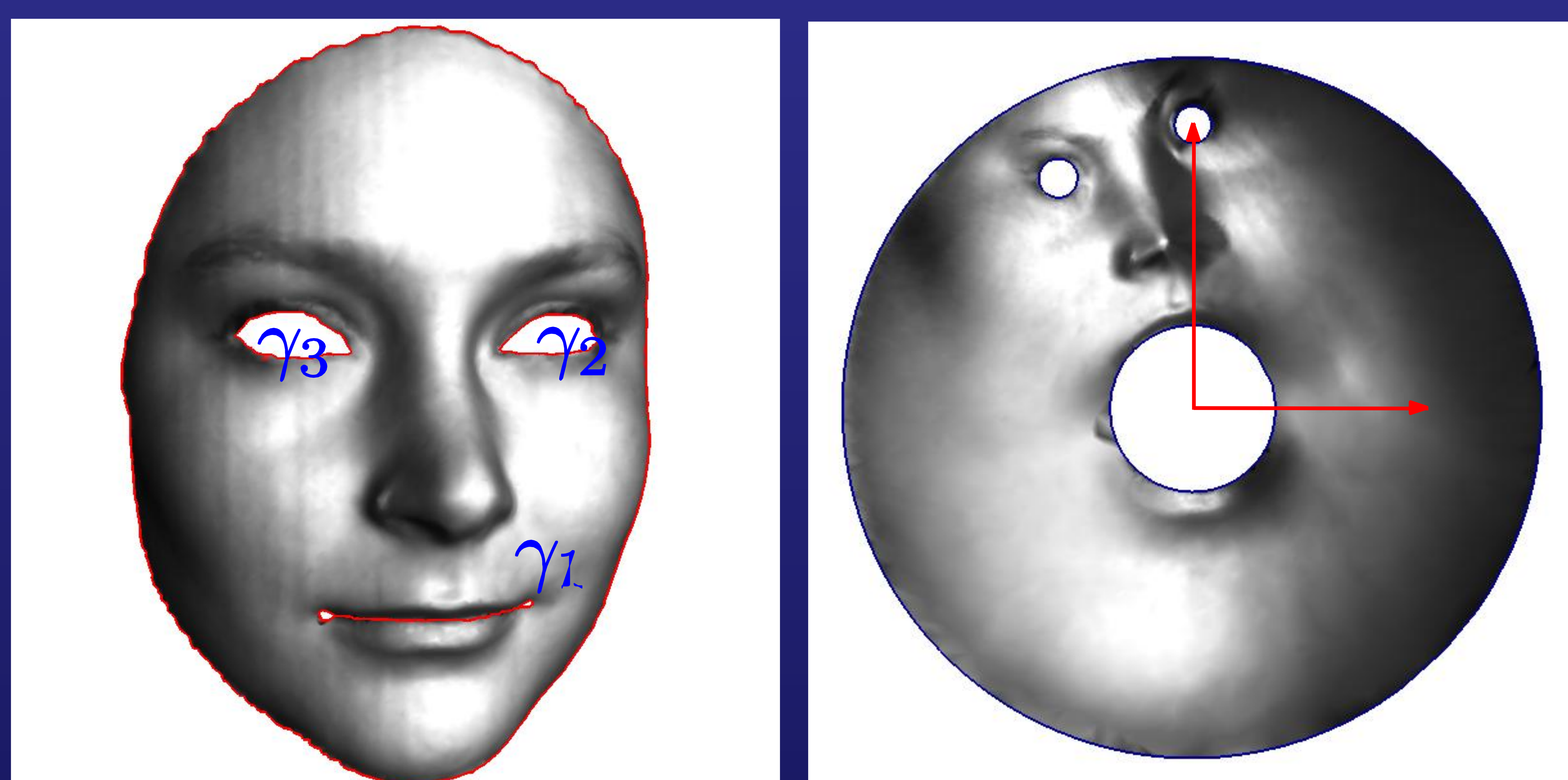
Conformal Maps of Koebe's M.

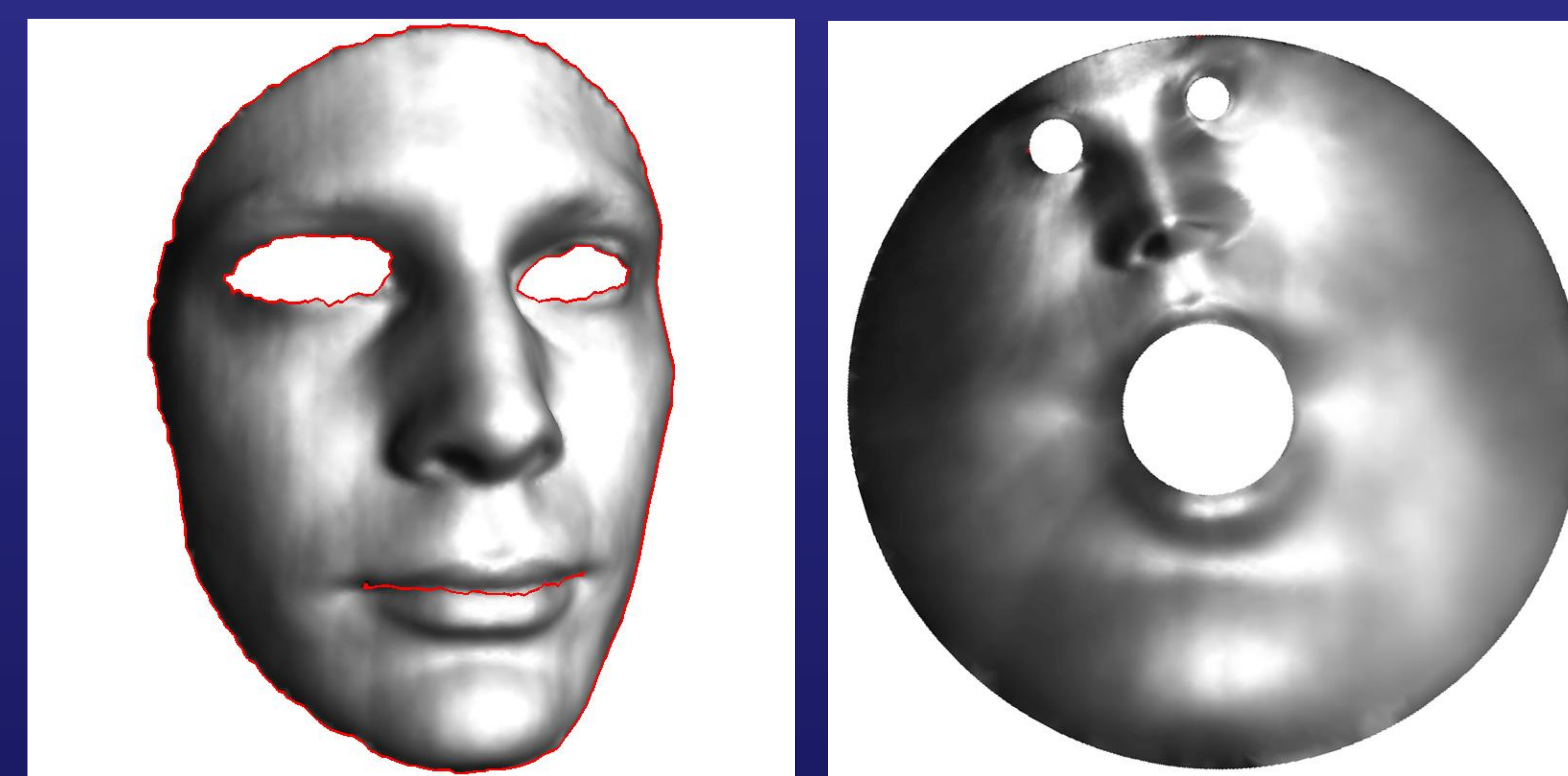
- 16 boundaries



Application

- ***Shape Analysis by Conformal Modules* [Zeng et al. 2009]**





Mod = (0.238, 0.809, 0.053, 0.657, -0.385, 0.055)

Mod = (0.234, 0.833, 0.057, 0.708, -0.411, 0.073)

Mod = (*r*1, *y*2, *r*2, *x*3, *y*3, *r*3)

The L_2 (Euclidean) distance is 0.064952.

Summary

- A practical algorithm to explicitly construct *conformal mappings* for *multiply connected domains*.
- *Generalized Koebe's Method*
  - General for *multiply connected Domains/Surfaces*.
  - *Quadratically* faster than conventional Koebe's method.
  - Rigorous *proof for exponential converging rate analysis*.
  - Efficient from *iterations of linear steps*.
  - Intrinsic to *conformal structure of surface*.

Generalized Koebe's Method for Conformal Mapping Multiply Connected Domains

Questions?

Thanks!

Wei Zeng [1] Xiaotian Yin [1] Min Zhang [1] Feng Luo [1] Xianfeng Gu [1]
[1] *State University of New York at Stony Brook* [2] *Rutgers University*