Riemann Uniformization using Ricci Flow Method

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Research

Phenomena | Explanation | Theory | Method | Technology | Product

Purposes

Concepts, theories and algorithms for computing uniformization metrics using Ricci flow method.

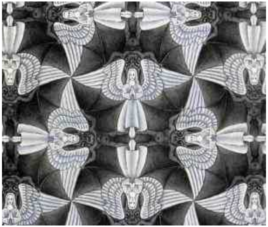
Concepts

Riemann Uniformization theorem, uniformization metric, Ricci flow, Fuchsian group

Algorithms to be covered

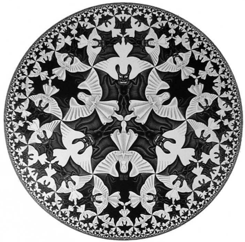
- Computing Euclidean Ricci Flow.
- Computing Hyperbolic Ricci Flow.
- Computing Fuchsian group.
- Computing Teichüller coordinates.

M.C.Esher's art works: Angels and Devils



Regular division
of the plane

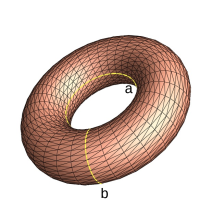
Sphere with Angels
and Devils

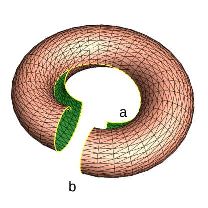


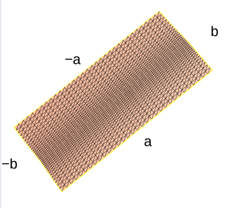
Circle limit IV
Heaven and Hell

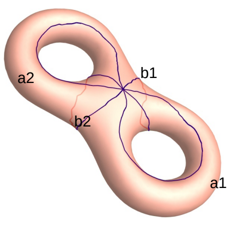
Universal Covering Space

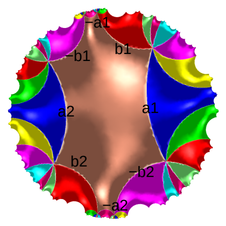
We can cut along some special curves of a surface and spread the surface on the plane or the disk.







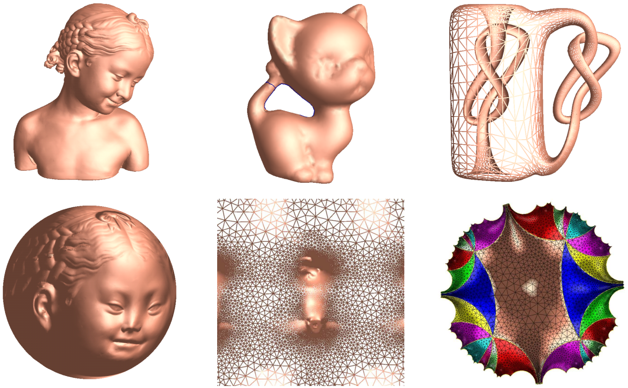




Uniformization

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2u}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical Euclidean Hyperbolic

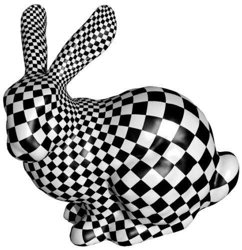
Conformal Metric

Definition

Suppose Σ is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. $e^{2\lambda}$ is called the conformal factor.



Angles are invariant measured by conformal metrics.

Conformal Metrics

Given a surface Σ with a Riemannian metric $\mathbf{g}$, find a function $u : \Sigma \to \mathbb{R}$, such that $e^{2u}\mathbf{g}$ is one of the followings:

1. Uniform flat metric
$$\bar{K} \equiv 0, \forall p \in \Sigma/\partial\Sigma \text{ and } \bar{k}_g \equiv const, \forall p \in \partial\Sigma$$
2. Uniformization metric
$$\bar{K} \equiv const, \forall p \in \Sigma/\partial\Sigma \text{ and } \bar{k}_g \equiv 0, \forall p \in \partial\Sigma$$
3. with prescribed curvature

The tool to calculate the above metrics is Ricci flow.

Definition (Surface Ricci Flow)

A closed surface with a Riemannian metric **g**, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -Kg_{ij}.$$

If the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant every where.

Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Theorem (Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.

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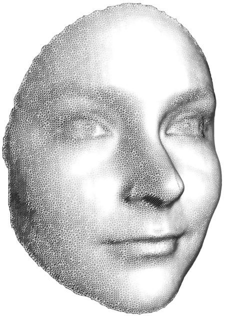
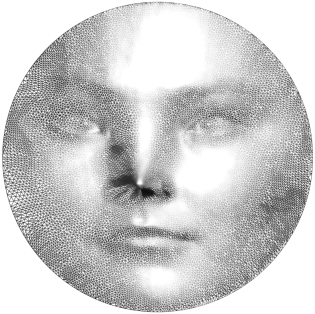
Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \to \mathbb{R}^1$, satisfies triangular inequality.

A mesh has infinite metrics.



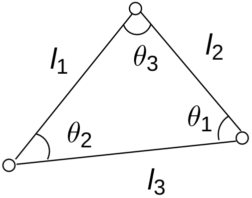


Discrete Metrics

Metric

- Discrete Metric: $l : E = \{all\ edges\} \to \mathbb{R}^1$, satisfies triangular inequality.
- Metrics determine curvatures by cosine law.

$$\cos\theta_i = \frac{l_j^2 + l_k^2 - l_i^2}{2 l_j l_k}, l \neq j \neq k \neq i$$



Derivative Cosine Law

Theorem (Derivative Cosine Law)

Consider an Euclidean triangle $\theta_i = \theta_i(l_1, l_2, l_3)$, $i \neq j \neq k \neq i$, *then*

$$\frac{1}{\sin\theta_i}\frac{\partial\theta_i}{\partial l_j} = \frac{1}{\sin\theta_j}\frac{\partial\theta_j}{\partial l_i}$$

Metric Space

The space of all Euclidean metric on a triangle

$$E(2) = \{(l_1, l_2, l_3) | l_i + l_j > l_k\}$$

Energy on metric space

Suppose we have a differential one-form

$$\omega = \sum \log \tan \frac{\theta_i}{2} dl_i,$$

then $d\omega = 0$, ω is a closed one-form,therefore

$$F(l_1, l_2, l_3) = \int_{(1,1,1)}^{(l_1,l_2,l_3)} \omega$$

is a well defined energy on metrics.

Convexity of the energy

The derivative is

$$\frac{\partial F}{\partial l_i} = ln \tan \frac{\theta_i}{2}$$

the Hessian is

$$\frac{\partial^2 F}{\partial l_i \partial l_j} = [\frac{1}{\sin \theta_i} \frac{\partial \theta_i}{\partial \theta_j}]_{3\times 3}$$

semi-positive definite. The energy is convex.

Metric Energy on Mesh

For a triangular mesh $(\Sigma, T, \mathbf{l})$, where T is the triangulation, l is the metric (edge length), define its energy $E(\mathbf{l})$, the sum of energy of its triangles

$$E(\mathbf{l}) = \sum_{[i,j,k] \in T} F(l_i, l_j, l_k)$$

then

- $E(\mathbf{l})$ is convex.
- $$\frac{\partial E}{\partial l_i} = ln \tan \frac{\alpha}{2} \tan \frac{\beta}{2}$$
  α, β are opposite to edge e_i.

Metric Energy on Mesh

Let $\psi : \{all\ edges\} \to \mathbb{R}$,

$$\psi(e) = \tan\frac{\alpha}{2}\tan\beta 2$$

called edge invariants. Then

$$\nabla E = (ln\psi(e_1), ln\psi(e_2), \cdots, ln\psi(e_n)).$$

Theorem

Suppose $\phi : \Omega \to \mathbb{R}$ is strictly convex, then $x \to \nabla\phi$ is a one to one map.

Theorem (Edge invariants)

A triangular mesh is determined upto isometry and scaling by its edge invariant, $\psi : \{all\ edges\} \to \mathbb{R}$.

Discrete Curvature

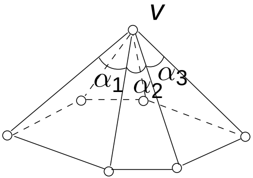
Definition (Discrete Curvature)

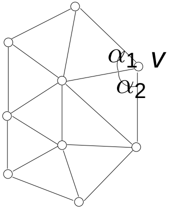
Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$





Relation between Metrics and Curvatures

Metrics vs. Curvatures

- All metrics for a mesh $\mathbf{L}(\Sigma)$ form a convex polytope.
- All admissible curvature configurations for a mesh $\mathbf{K}(\Sigma)$ also form a convex polytope.
- The mapping from the metrics to the curvatures

$$\Phi : \mathbf{L}(\Sigma) \to \mathbf{K}(\Sigma),$$

is not one to one.

Discrete Prescribed Curvature

Theorem

Given a prescribed curvature function K, $\phi^{-1}(K)$ is a $|E| - |V|$ dimensional manifold.

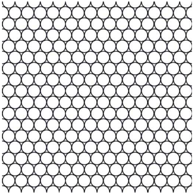
Theorem (Prescribed Curvature)

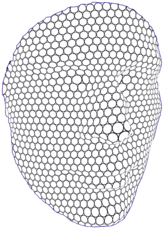
The mapping from a conformal class of metrics to the curvatures is a homeomorphism.

Conformal metric deformation

Conformal maps Properties

- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.





Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

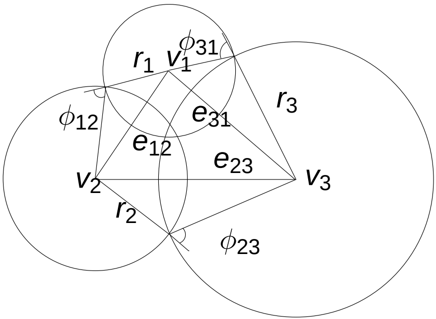
CP Metric

We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\Phi_{ij}$$

CP Metric (Σ, Γ, Φ), Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\phi_{ij} | \forall e_{ij}\}$$



Definition (Conformal Circle Packing Metrics)

Two circle packing metrics $\{\Sigma, \Phi_1, \Gamma_1\}$ and $\{\Sigma, \Phi_2, \Gamma_2\}$ are conformal equivalent, if

- The radii of circles are different, $\Gamma_1 \neq \Gamma_2$.
- The intersection angles are same, $\Phi_1 \equiv \Phi_2$.

In practice, the circle radii and intersection angles are optimized to approximate the induced Euclidean metric of the mesh as close as possible.

Definition (Discrete Ricci flow)

A mesh Σ with a circle packing metric $\{\Sigma, \Gamma, \Phi\}$, where $\Gamma = \{\gamma_i, v_i \in V\}$ are the vertex radii, $\Phi = \{\Phi_{ij}, e_{ij} \in E\}$ are the angles associated with each edge, the discrete Ricci flow on Σ is defined as

$$\frac{d\gamma_i}{dt} = (\bar{K}_i - K_i)\gamma_i,$$

where $\bar{K}_i$ are the target curvatures on vertices. If $\bar{K}_i \equiv 0$, the flow with normalized total area leads to a metric with constant Gaussian curvature.

Idea

Metric deformation is driven by curvature.

Theorem (Chow and Luo 2002)

A discrete Euclidean Ricci flow $\{\Sigma, \Gamma, \Phi\} \rightarrow \{M, \bar{\Gamma}, \Phi\}$ *converges.*

$$|K_i(t) - \bar{K}_i| < c_1 e^{-c_2 t},$$

and

$$|\gamma_i(t) - \bar{\gamma}_i| < c_1 e^{-c_2 t},$$

where c_1, c_2 *are positive numbers.*

Definition

Let $u_i = ln\gamma_i$, the Ricci energy is defined as

$$f(\mathbf{u}) = \int_{\mathbf{u}_0}^{\mathbf{u}} \sum_{i=1}^{n} (K_i - \bar{K}_i) du_i,$$

where $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, $\mathbf{u}_0 = (0, 0, \cdots, 0)$.

Derivative Euclidean Cosine Law

Theorem (Ricci Energy)

Euclidean Ricci energy is Well defined and convex, namely, there exists a unique global minimum.

Proof.

In an Euclidean triangle, with angles $(\theta_1, \theta_2, \theta_3)$ and radius $(\gamma_1, \gamma_2, \gamma_3)$, let $u_i = ln\gamma_i$, according to Euclidean cosine law,

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.$$

Therefore $\omega = \sum \theta_i du_i$ is a closed 1-form. The Euclidean Ricci energy is well defined. Direct computation verifies that Hessian matrix is positive definite. □

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Newton's method for Euclidean Ricci energy

Gradient descent Method

Ricci flow is the gradient descent method for minimizing Ricci energy,

$$\nabla f = (K_1 - \bar{K}_1, K_2 - \bar{K}_2, \cdots, K_n - \bar{K}_n).$$

Newton's method

The Hessian matrix of Ricci energy is

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial K_i}{\partial u_j}.$$

Newton's method can be applied directly.

Algorithm : uniform flat metric for closed surfaces

Ricci Flow for Uniform Flat Metric

Suppose Σ is a closed genus one mesh,

1. Compute the circle packing metric (Γ, Φ).
2. Set the target curvature to be zero for each vertex

$$\bar{K}_i \equiv 0, \forall v_i \in V$$

3. Minimize the Euclidean Ricci energy using Newton's method to get the target radii $\bar{\Gamma}$.
4. Compute the target flat metric.

Algorithm : uniform flat metric for open surfaces

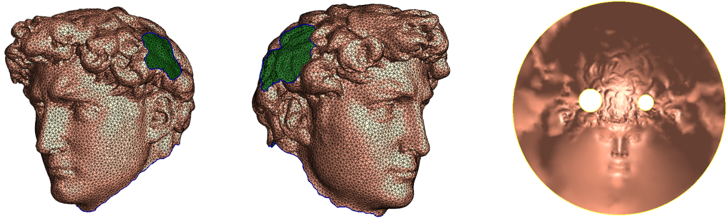
Given a surface Σ with genus g and b boundaries, then it Euler number is

$$\chi(\Sigma) = 2 - 2g - b.$$

Suppose the boundary of Σ is a set of closed curves

$$\partial\Sigma = C_1 \cup C_2 \cup C_3 \cdots C_b.$$

The total curvature for each C_i is denoted as $2m_i\pi, m_i \in \mathbb{Z}$, and $\sum_{i=1}^{b} m_i = \chi(\Sigma)$. The target curvature for interior vertices are zeros



Algorithm : uniform flat metric for open surfaces

Euclidean Ricci flow for open surfaces

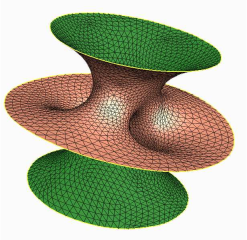
- Use Newton's method to minimize the Ricci energy to update the metric.
- Adjust the boundary vertex curvature to be proportional to the ratio between the current lengths of the adjacent edges and the current total length of the boundary component.
- Repeat until the process converges.

Algorithm : Flatten a mesh with a uniform flat metric

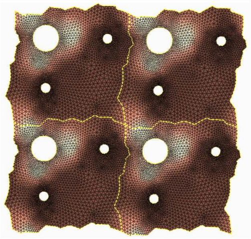
Embedding

1. Determine the planar shape of each triangle using 3 edge lengths.
2. Glue all triangles on the plane along their common edges by rigid motions. Because the metric is flat, the gluing process is coherent and results in a planar embedding.

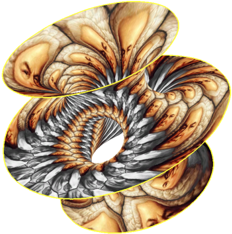
Euclidean Uniform Flat Metric



original surface
genus 1, 3 boundaries

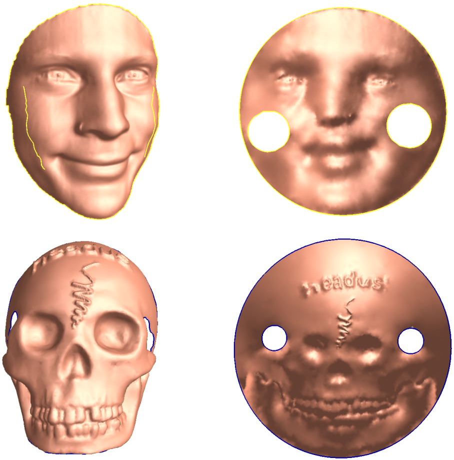


universal cover
embedded in $\mathbb{R}^2$

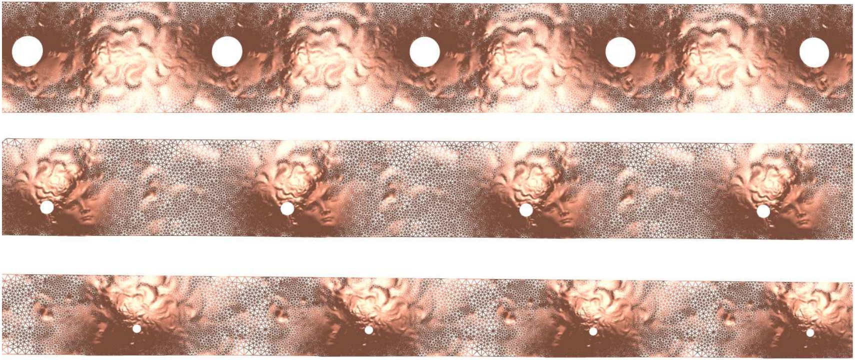


texture mapping

Euclidean Uniform Flat Metric

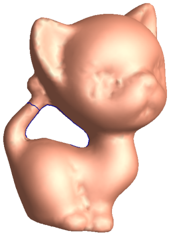


Euclidean Uniform Flat Metric

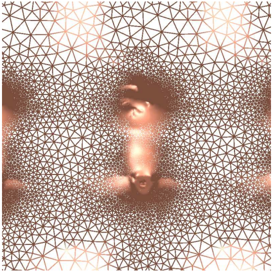


Different boundaries are mapped to straight lines.

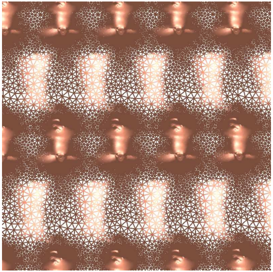
Euclidean Uniform Flat Metric



original surface



fundamental domain



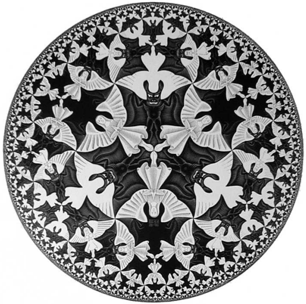
universal cover

Conformal Model : Poincaré Disk

Poincaré disk

A unit disk $|z| < 1$ with the Riemannian metric

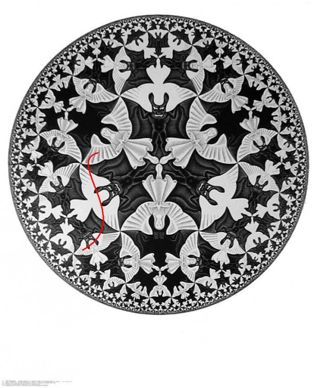
$$ds^2 = \frac{4dzd\bar{z}}{(1 - \bar{z}z)^2}.$$



Poincaré disk

The rigid motion is the Möbius transformation

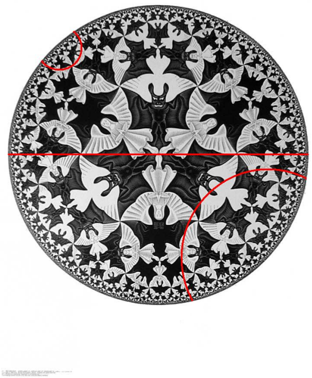
$$e^{i\theta}\frac{z-z_0}{1-\bar{z}_0 z}.$$



Conformal Model : Poincaré Disk

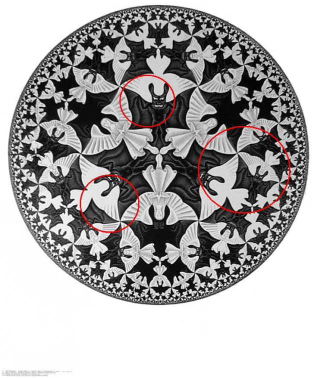
Poincaré disk

The hyperbolic line through two point z_0, z_1 is the circular arc through z_0, z_1 and perpendicular to the boundary circle $|z| = 1$.



Poincaré disk

A hyperbolic circle (c,γ) on Poincare disk is also an Euclidean circle (C,R) on the plane, such that $\mathbf{C} = \frac{2-2\mu^2}{1-\mu^2|\mathbf{c}|^2}$, $R^2 = |\mathbf{C}|^2 - \frac{|\mathbf{c}|^2-\mu^2}{1-\mu^2|\mathbf{c}|^2}, \mu = \frac{e^r-1}{e^r+1}$.



Discrete Hyperbolic Ricci Flow

Definition (Discrete Hyperbolic Ricci Flow)

Let

$$u_i = ln \tanh \frac{\gamma_i}{2},$$

Discrete hyperbolic Ricci flow for a mesh Σ is

$$\frac{du_i}{dt} = \bar{K}_i - K_i, \bar{K}_i \equiv 0,$$

the Euler number of Σ is negative, $\chi(\Sigma) < 0$.

Discrete Hyperbolic Ricci flow

Theorem (Discrete Hyperbolic Ricci flow, Chow and Luo 2002)

A hyperbolic discrete Ricci flow $(M,\Gamma,\Phi) \to (M,\bar{\Gamma},\Phi)$ *converges,*

$$|K_i(t) - \bar{K}_i| < c_1 e^{-c_2 t},$$

and

$$|\gamma_i(t) - \bar{\gamma}_i| < c_1 e^{-c_2 t},$$

where c_1, c_2 *are positive numbers.*

Definition (Discrete Hyperbolic Ricci Energy)

The discrete Hyperbolic Ricci energy is defined as

$$f(\mathbf{u}) = \int_{\mathbf{u}_0}^{\mathbf{u}} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i.$$

Discrete hyperbolic Ricci flow is the gradient descendent method to minimize the discrete hyperbolic ricci energy.

Theorem (Hyperbolic Discrete Ricci Energy)

Discrete hyperbolic Ricci energy is well defined and convex, namely, there exists a unique global minimum.

Proof.

In a hyperbolic triangle, with angles $(\theta_1, \theta_2, \theta_3)$ and radius $(\gamma_1, \gamma_2, \gamma_3)$, $u_i = ln \tanh \frac{\gamma_i}{2}$, according to hyperbolic cosine law,

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.$$

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Derivative hyperbolic Cosine Law

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Therefore $\omega = \sum \theta_i du_i$ is a closed 1-form. The hyperbolic Ricci energy is convex. Direct computation verifies the Hessian matrix is positive definite. □

Algorithm: Computing Hyperbolic uniformization metric

Hyperbolic Ricci Energy Optimization

1. Set target curvature $K(v_i) \equiv 0$.
2. Optimize the hyperbolic Ricci energy using Newton's method, with the constraint the total area is preserved.

Flattening Mesh in Hyperbolic Space

1. Determine the shape of each triangle.
2. Glue the hyperbolic triangles coherently by Möbius transformation.

Key: all computations use hyperbolic geometry.

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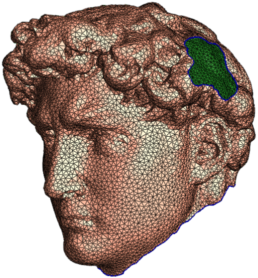
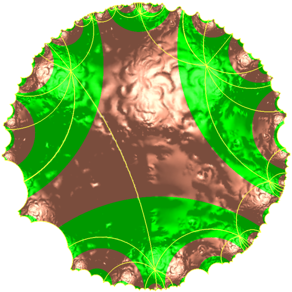
1. Determine the shape of each triangle.
2. Glue the hyperbolic triangles coherently by Möbius transformation.

Key: all computations use hyperbolic geometry.

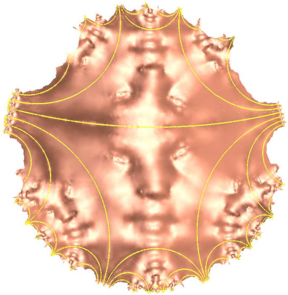
Hyperbolic Uniformization Metric





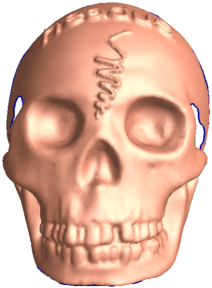
Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.

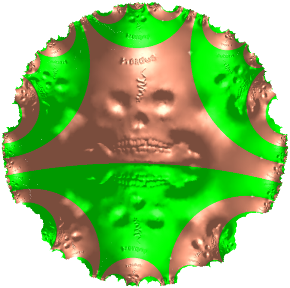
Hyperbolic Uniformization Metric



Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.

Hyperbolic Uniformization Metric





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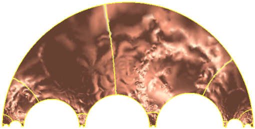
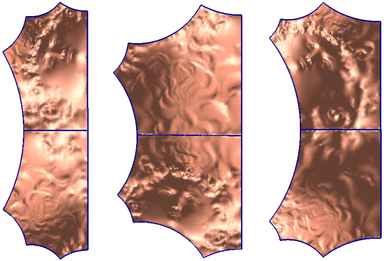
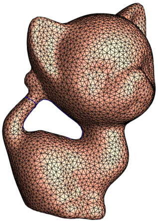
Hyperbolic Uniformization Metric

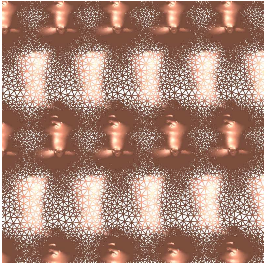




Embedding in the upper half plane hyperbolic space model. Different period embedded in the hyperbolic space. The boundaries are mapped to hyperbolic lines.

Universal Covering Space and Deck Transformation





Universal Cover

A pair $(\bar{\Sigma},\pi)$ is a universal cover of a surface Σ, if

- Surface $\bar{\Sigma}$ is simply connected.
- Projection $\pi:\bar{\Sigma}\to\Sigma$ is a local homeomorphism.

Deck Transformation

A transformation $\phi:\bar{\Sigma}\to\bar{\Sigma}$ is a deck transformation, if

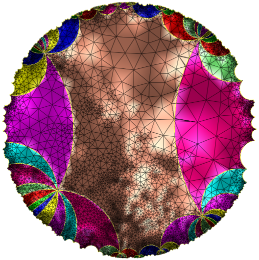
$$\pi=\pi\circ\phi.$$

A deck transformation maps one period to another.

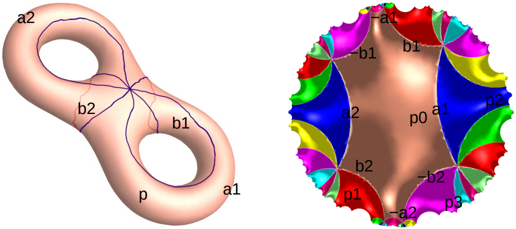
Definition (Funchsian Group)

Suppose Σ is a surface, $\mathbf{g}$ is its uniformization metric, $(\bar{\Sigma}, \pi)$ is the universal cover of Σ. $\mathbf{g}$ is also the uniformization metric of $\bar{\Sigma}$. A deck transformation of $(\bar{\Sigma}, \mathbf{g})$ is a Möbius transformation. All deck transformations form the Fuchsian group of Σ.

Fuchsian group indicates the intrinsic symmetry of the surface.



Fuchsian Group



The Fuchsian group is isomorphic to the fundamental group

	$e^{i\theta}$	z_0
a_1	$-0.631374 + i0.775478$	$+0.730593 + i0.574094$
b_1	$+0.035487 - i0.999370$	$+0.185274 - i0.945890$
a_2	$-0.473156 + i0.880978$	$-0.798610 - i0.411091$
b_2	$-0.044416 - i0.999013$	$+0.035502 + i0.964858$

Klein Model

Another Hyperbolic space model is Klein Model, suppose **s**, **t** are two points on the unit disk, the distance is

$$d(s,t) = arccosh \frac{1 - \mathbf{s}\cdot\mathbf{t}}{\sqrt{(1-\mathbf{s}\cdot\mathbf{s})(1-\mathbf{t}\cdot\mathbf{t})}}$$

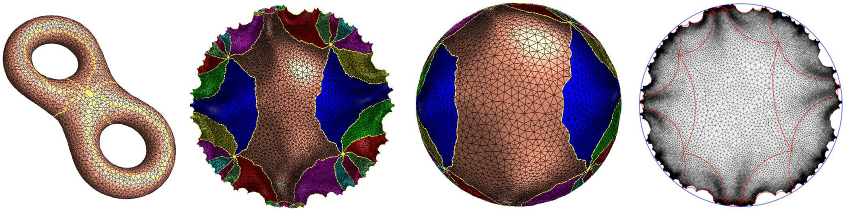
Poincaré vs. Klein Model

From Poincaré model to Klien model is straight froward

$$\beta(z) = \frac{2z}{1+\bar{z}z}, \beta^{-1}(z) = \frac{1-\sqrt{1-\bar{z}z}}{\bar{z}z},$$

Assume ϕ is a Möbius transformation, then transition maps $\beta\circ\phi\circ\beta^{-1}$ are real projective.

Hyperbolic and Real Projective Structure

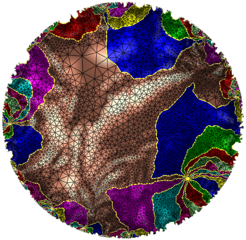


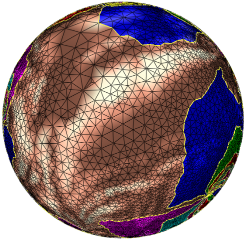
Real projective structure

The embedding of the universal cover in the Poincaré disk is converted to the embedding in the Klein model, which induces a real projective atlas of the surface.

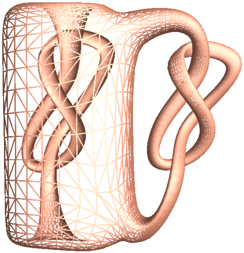
Hyperbolic and Real Projective Structure


Surface

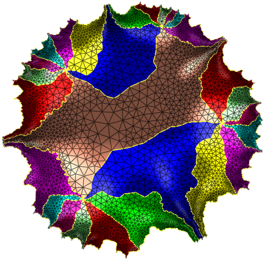

Hyperbolic Structure


Projective Structure

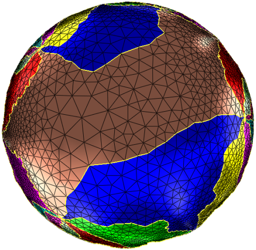
Hyperbolic and Real Projective Structure



Surface, courtesy of Cindy Grimm

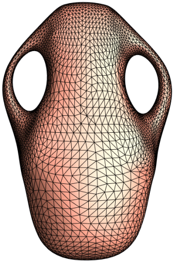


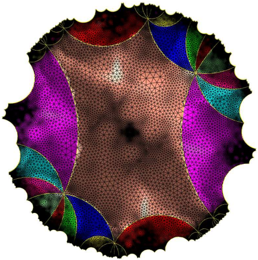
Hyperbolic Structure

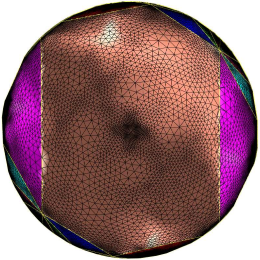


Projective Structure

Hyperbolic and Real Projective Structure

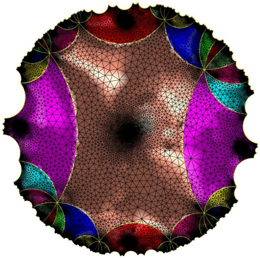

Surface


Hyperbolic Structure

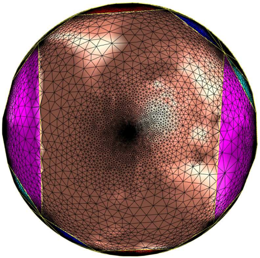

Projective Structure

Hyperbolic and Real Projective Structure

Surface

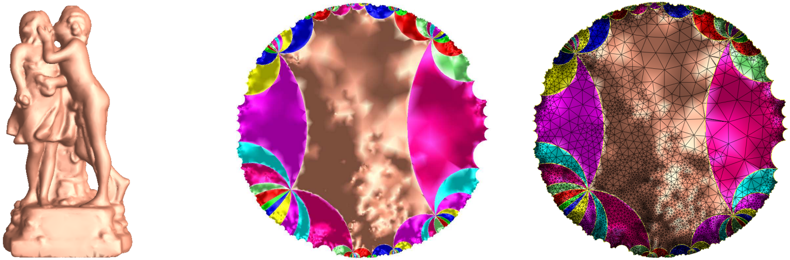


Hyperbolic Structure



Projective Structure

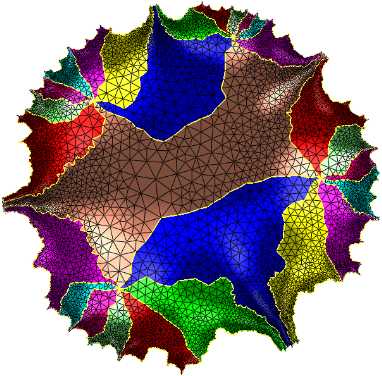
Hyperbolic Uniformization Metric



Challenges

Challenges

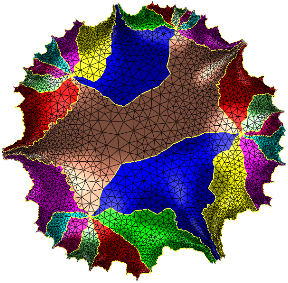
- Intrinsically nonlinear method.
- Intrinsically the conformal factor may be exponential.
- Determine the optimal initial circle packing metric.
- Embed universal cover in the Poincaré disk.



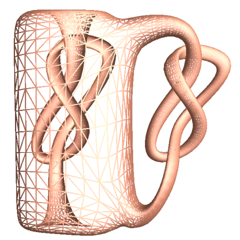
Future Directions

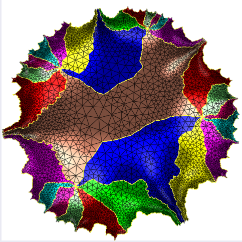
Future Works

- Design spline schemes based on real projective geometry.
- Hirearchical approach for Ricci energy optimization.
- Surface classification using Fuchsian group.
- Generalize planar geometric algorithms to surface domains using geometric structures.
- Ricci flow on 3-manifolds.



For more information, please email to gu@cs.sunysb.edu.





Thank you!