

Analysis of Aloha and Slotted Aloha

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1 Aloha and Slotted Aloha

Aloha is a time-division multiple access scheme. A user, whenever has data to transmit, immediately transmits, irrespective of whether the other users use the channel or not. In Slotted Aloha, time is divided into time slots. A user with data will transmit at the beginning of the next time slot.

In this note we analyze the throughput of the system. The throughput is defined as the ratio of the number of packets delivered successfully and the total number of packets possible (assuming perfect global coordination). The throughput measures the efficiency of the system.

1.1 Assumptions

In the following analysis we make the following assumptions.

- There are total number of N users in the system.
- All packets have length T .
- Each user transmits with probability p within a time period of T .

Now the average number of packets in the system within a period of time T is simply $\lambda = N \cdot p$. The average number of packets in the system within a period of time $2T$ is $\lambda' = 2N \cdot p$. The arrival of independent events with a rate of λ within unit time is modeled by the Poisson distribution:

$$P(k) = \frac{\lambda^k e^{-\lambda}}{k!},$$

in which λ is the *rate*, the average number of packet transmissions within unit time T , and $P(k)$ is the probability that k packet transmissions occur within unit time T .

1.2 Vulnerable periods

We use the term *vulnerable period* to denote the period of time that a packet can possibly collide with a given packet x . In Aloha system, the vulnerable period has length $2T$ — any packet that starts within time T from the starting point of x (either before or after) will collide with x . In slotted Aloha, two packets either collide completely or do not overlap at all. The vulnerable period is reduced to T — the slot before the transmission of packet x .

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1.3 Analysis of Aloha

We first focus on a specific packet x and calculate the probability that x is successfully delivered.

$$\begin{aligned} & \text{Prob}\{x \text{ is successfully delivered}\} \\ &= \text{Prob}\{\text{No other packets within the vulnerable period } 2T\} \end{aligned}$$

Now this probability can be calculated by the Poisson distribution with rate $\lambda' = 2Np$. Thus,

$$\begin{aligned} & \text{Prob}\{x \text{ is successfully delivered}\} \\ &= P(0) \\ &= \lambda'^0 e^{-\lambda'} / 0! \\ &= e^{-2Np}. \end{aligned}$$

To calculate the throughput of the system, we focus on a time period of length T , the best possible is to transmit one packet within this time period. In the Aloha system, there are about Np transmission attempts and each one has a probability of e^{-2Np} to go through. Thus the average number of packet that can successfully go through is

$$\text{Throughput} = Np \cdot e^{-2Np}.$$

We denote $R = Np$, now we calculate the optimal configuration to achieve the highest throughput. This is done by taking the derivative of the throughput function on R . By simple calculation, we have that when $R = 1/2$ the throughput achieves its highest point $1/(2e) \approx 0.18$. In other words, a system with Aloha uses at most 18% of the total resources.

Alternative analysis. From the above analysis, to get a message go through the channel, there must be exactly 1 message within a time interval of $2T$. The probability for this to happen, with a Poisson distribution of average message arrival rate of $\lambda = 2Np$ for a period of $2T$, is $P(1) = \lambda \cdot e^{-\lambda} = 2Npe^{-2Np}$. This is also the average number of messages that would go through within a time period of $2T$. Thus the throughput, i.e., the average number of messages that would go through within a time period of T , is Npe^{-2Np} .

We can also calculate the average number of attempts before a packet is successfully delivered. First we calculate the probability that a packet is successfully delivered after exactly k trials.

$$\begin{aligned} & \text{Prob}\{\text{A packet is successfully delivered after } k \text{ times}\} \\ &= \text{Prob}\{\text{The first } k-1 \text{ times failed and the last time is succesful}\} \\ &= (1 - e^{-2Np})^{k-1} \cdot e^{-2Np} \end{aligned}$$

Now we are ready to calculate the average number of trials to send a packet over.

$$\begin{aligned} & \text{The average number of trials to send over a packet} \\ &= \sum_{k=1}^{\infty} \text{Prob}\{\text{A packet is successfully delivered after } k \text{ times}\} \cdot k \\ &= \sum_{k=1}^{\infty} (1 - e^{-2Np})^{k-1} \cdot e^{-2Np} \cdot k \\ &= e^{2Np} \end{aligned}$$

Now take the optimal configuration with $R = Np = 1/2$, this means that the average number of trials, even in the optimal configuration, is about $e=2.7$.

1.4 Analysis of slotted Aloha

The analysis of the throughput in the case of slotted Aloha is almost the same as that for Aloha, except that the vulnerable period is shrunk from $2T$ to T . Thus the only change is the probability that a packet x is successfully delivered.

$$\begin{aligned} & \text{Prob}\{x \text{ is successfully delivered}\} \\ &= \text{Prob}\{\text{No other packets within the vulnerable period } T\} \\ &= e^{-Np} \end{aligned}$$

Thus,

$$\text{Throughput} = Np \cdot e^{-Np},$$

which achieves the highest point $1/e = 36\%$ when $R = Np = 1$.

Alternative analysis. Consider a particular time slot and a particular user, the chance that this user sends a message through during this slot is

$$p(1-p)^{N-1}.$$

The average number of messages that would go through during this slot is then

$$Np(1-p)^{N-1}.$$

To maximize the throughput, we choose $p = 1/N$. In this case, the throughput is

$$(1 - 1/N)^{N-1} = 1/e.$$