# The Emergence of Sparse Spanner and Greedy Well-Separated Pair Decomposition

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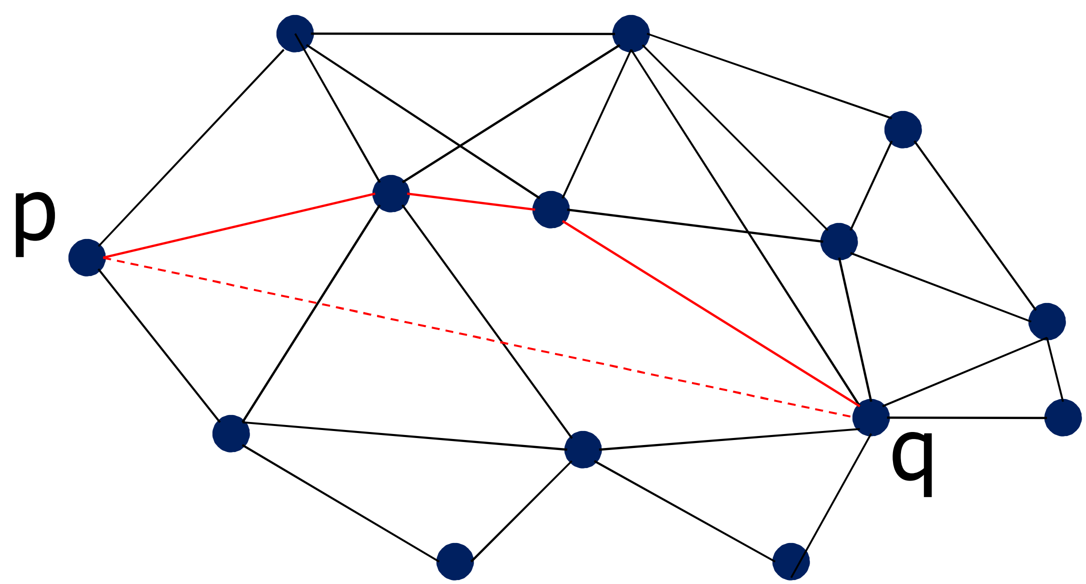
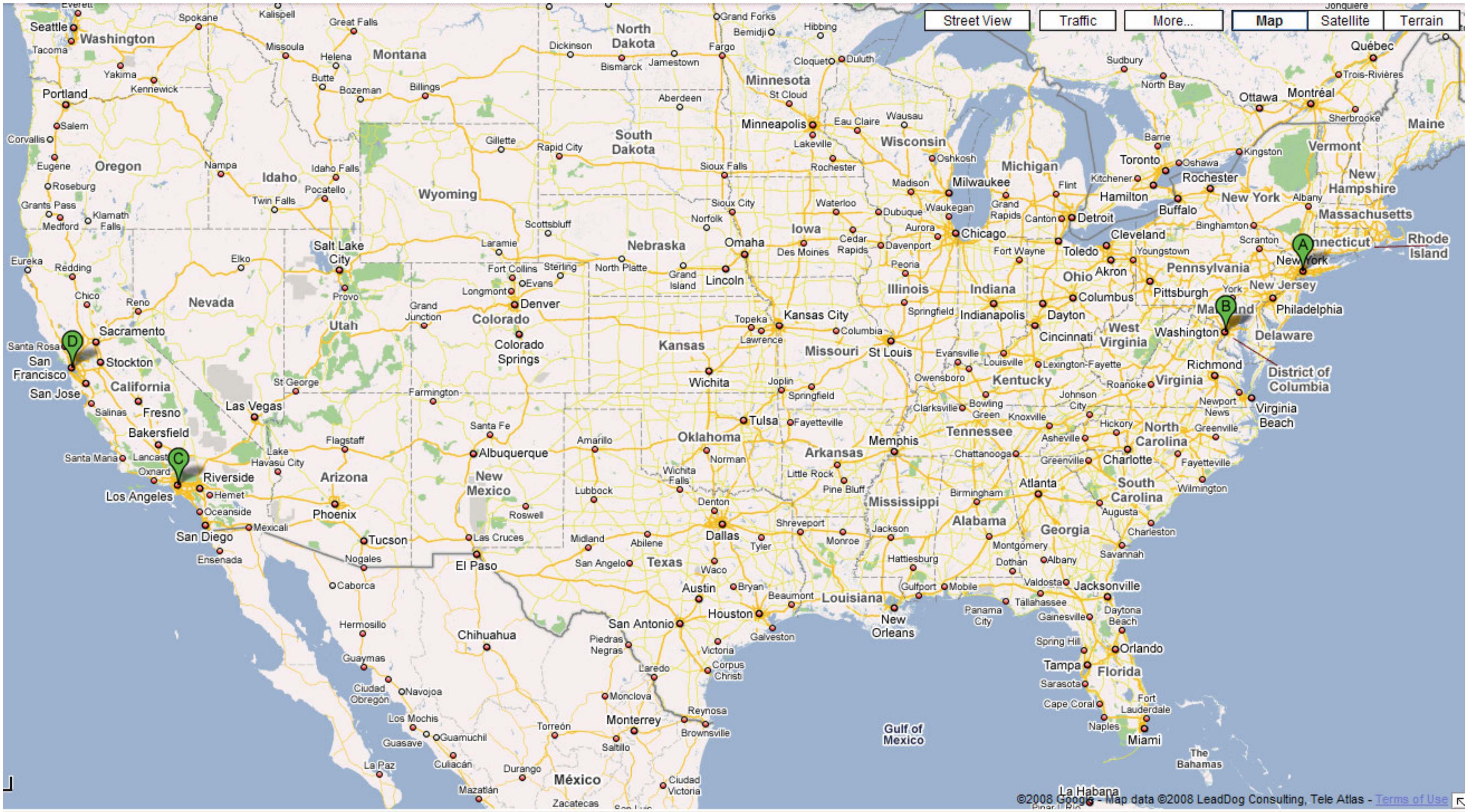
# t-Spanner

- Definition
  - A geometric graph G of a set of points P, the shortest path between any two points p,q in G has length at most t |pq|.



# Good Spanners in Reality



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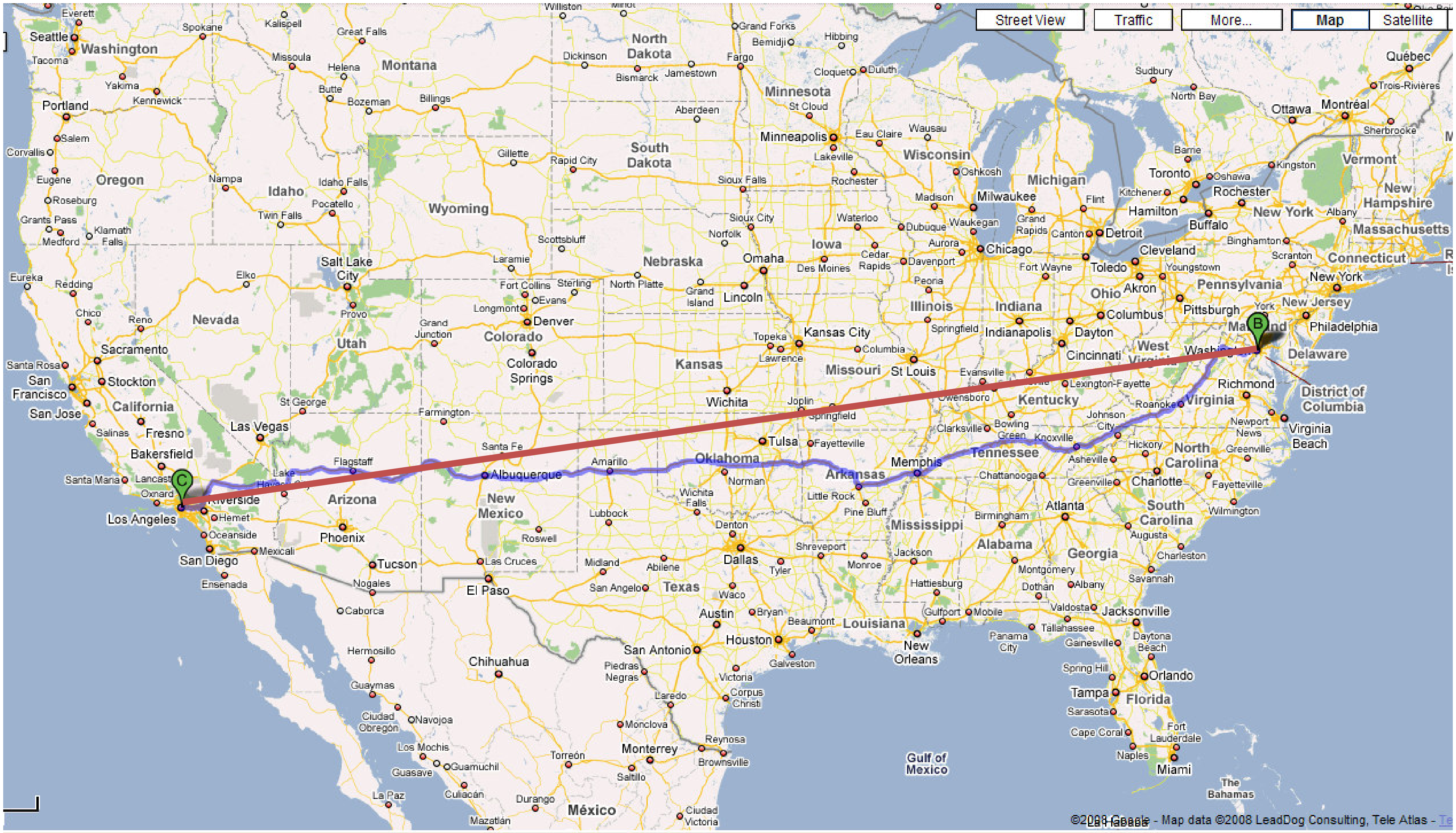


# Good Spanners in Reality

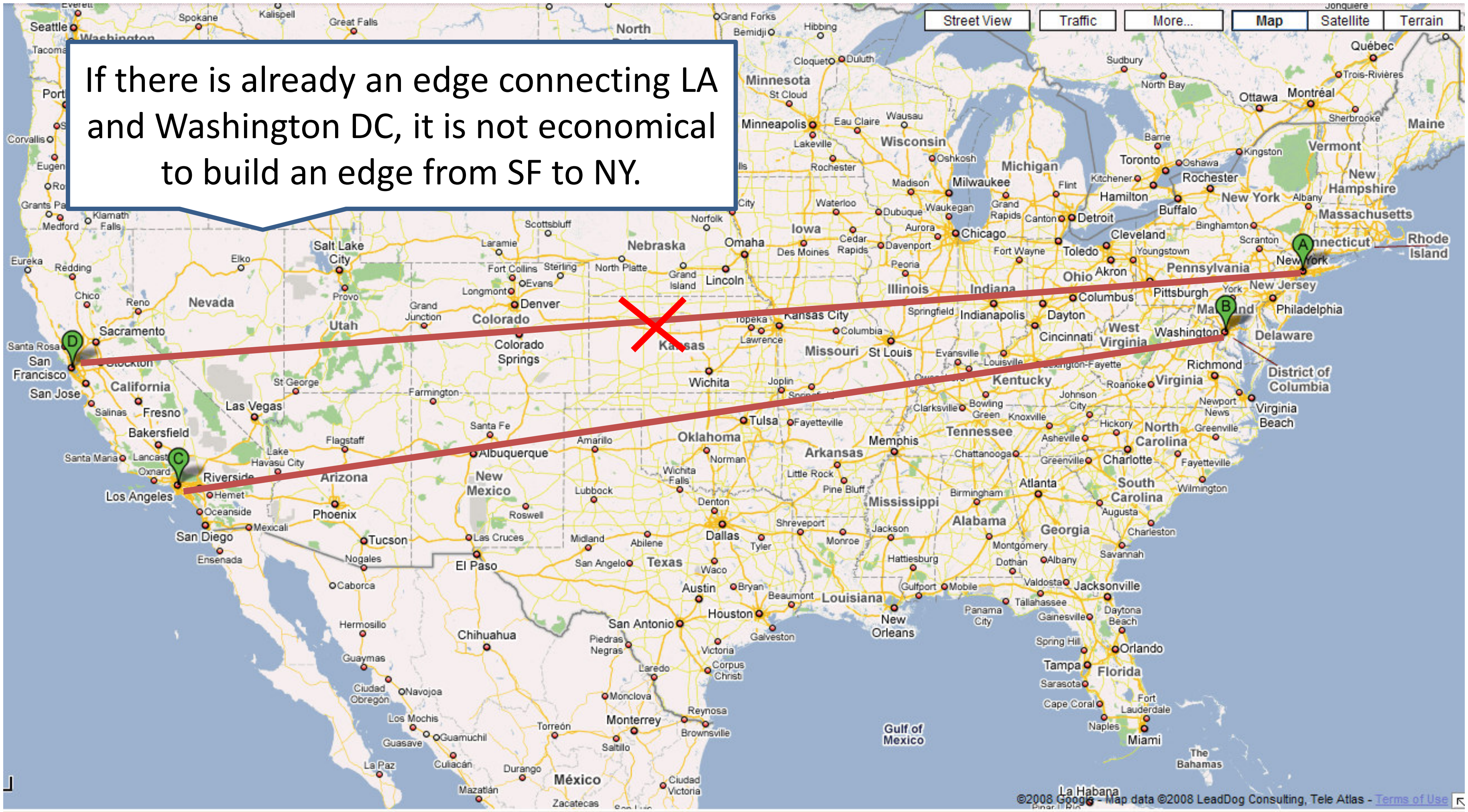
- There are many good spanners in reality
  - Road network
  - Flight network
  - Internet backbone.
- They are
  - Not build at the same time.
  - Not owned by the same authority.
- Q: how does a good spanner emerge?

# How does a sparse spanner emerge?

- A set of distributed agents in $R^2$ want to build a sparse spanner graph.
  - No global coordination.
  - Edges are not necessarily built all at the same time.

- A simple rule to tell whether an edge should be built or not?

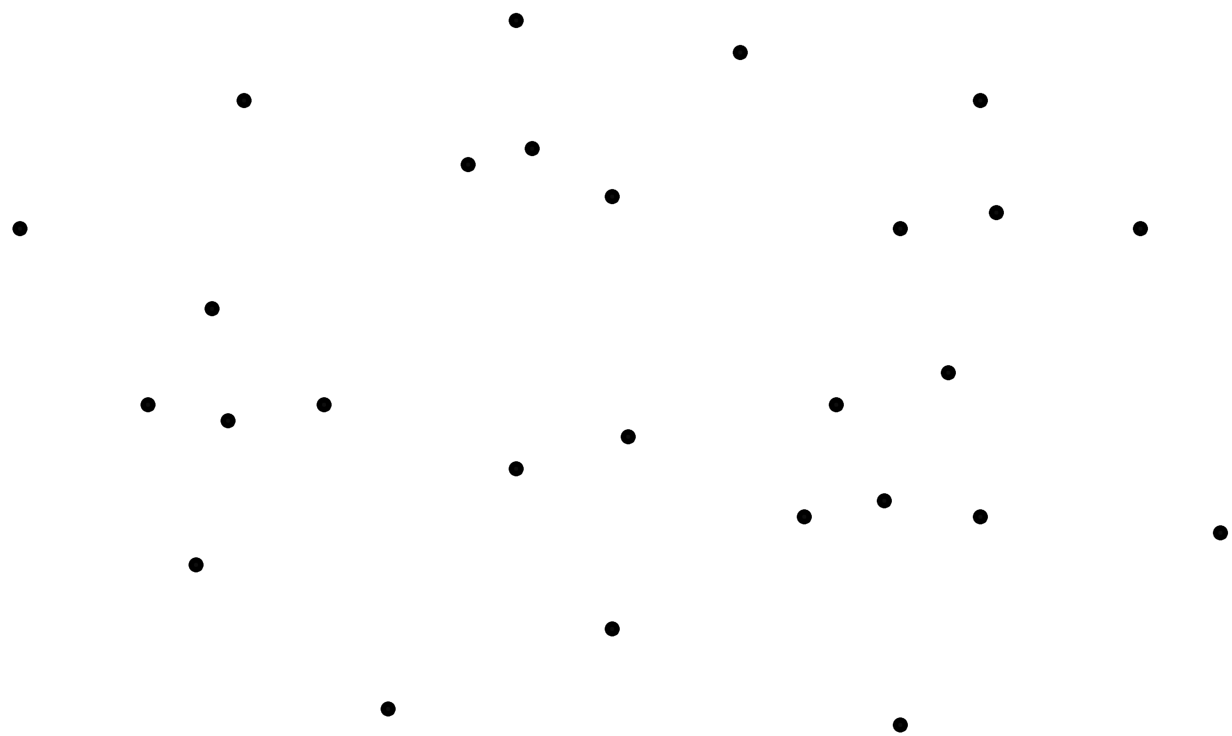
# Good Spanners in Reality



# Emergence of Sparse Spanner

- A de-centralized algorithm:
  - Each node **p** performs the following operation when considering build an edge to q:
    - Check where there is already an edge p'q' such that p and q are within distance **|p'q'|/(2(s+1))** from p', q' respectively.
    - If so, p does not build the edge to q
    - Otherwise, p will build the edge to q
  - Continue until all pairs are checked.
  - The orders of the edges checked do not matter.

# An Example



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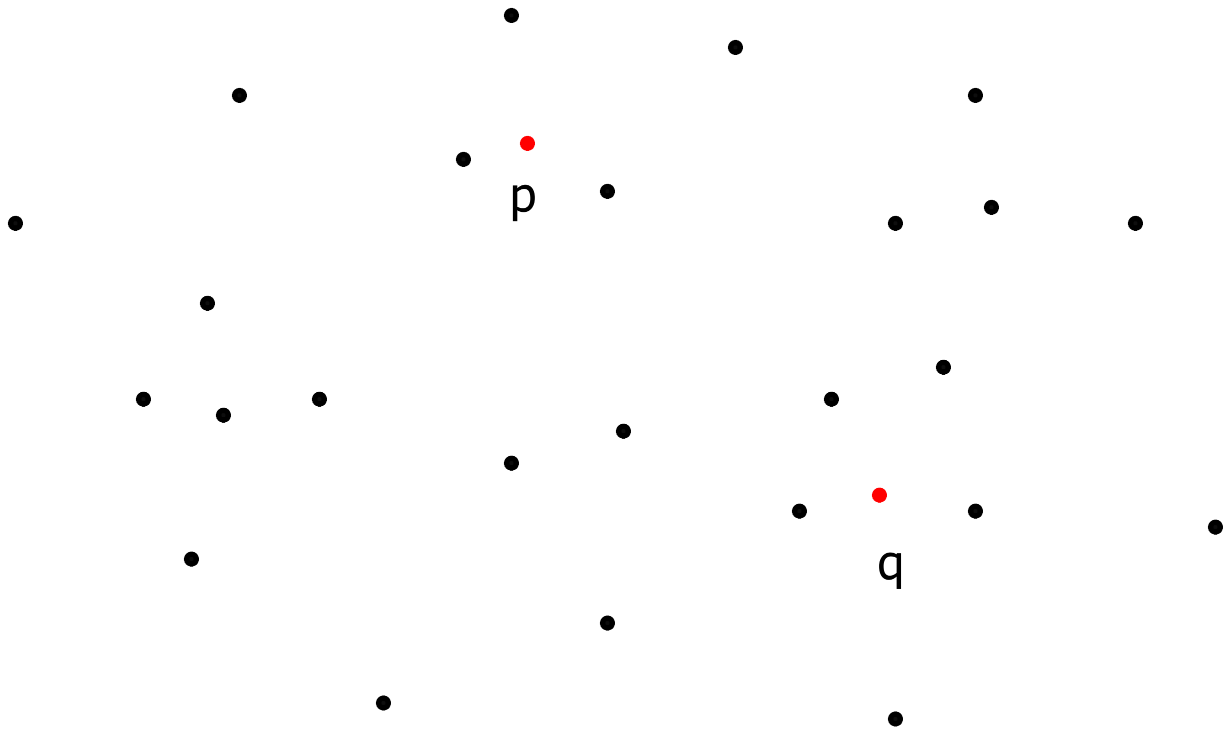

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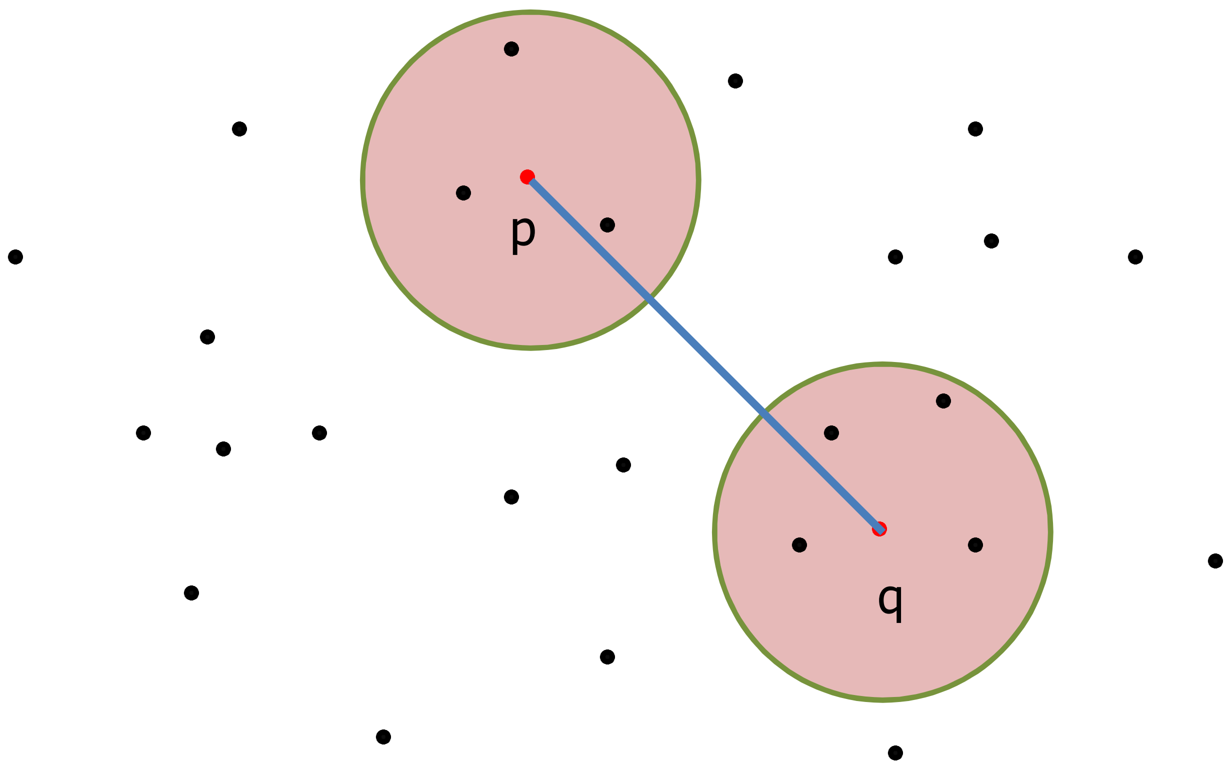

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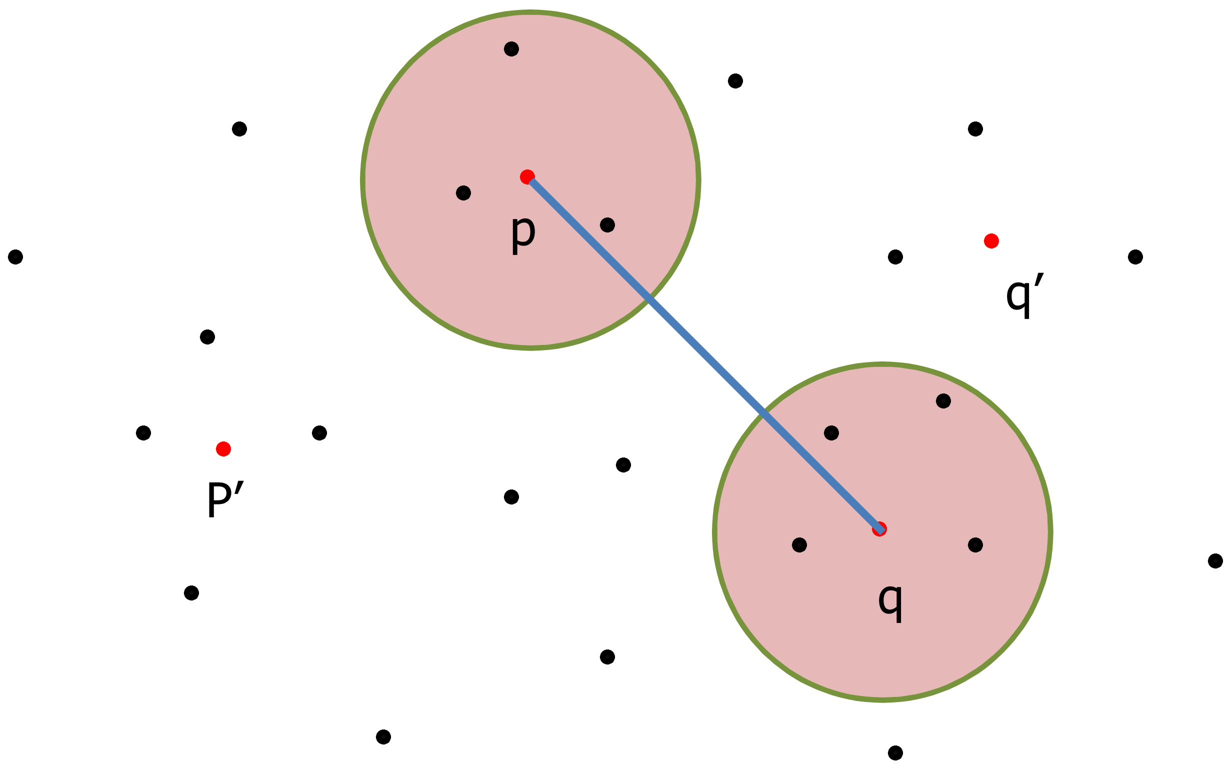

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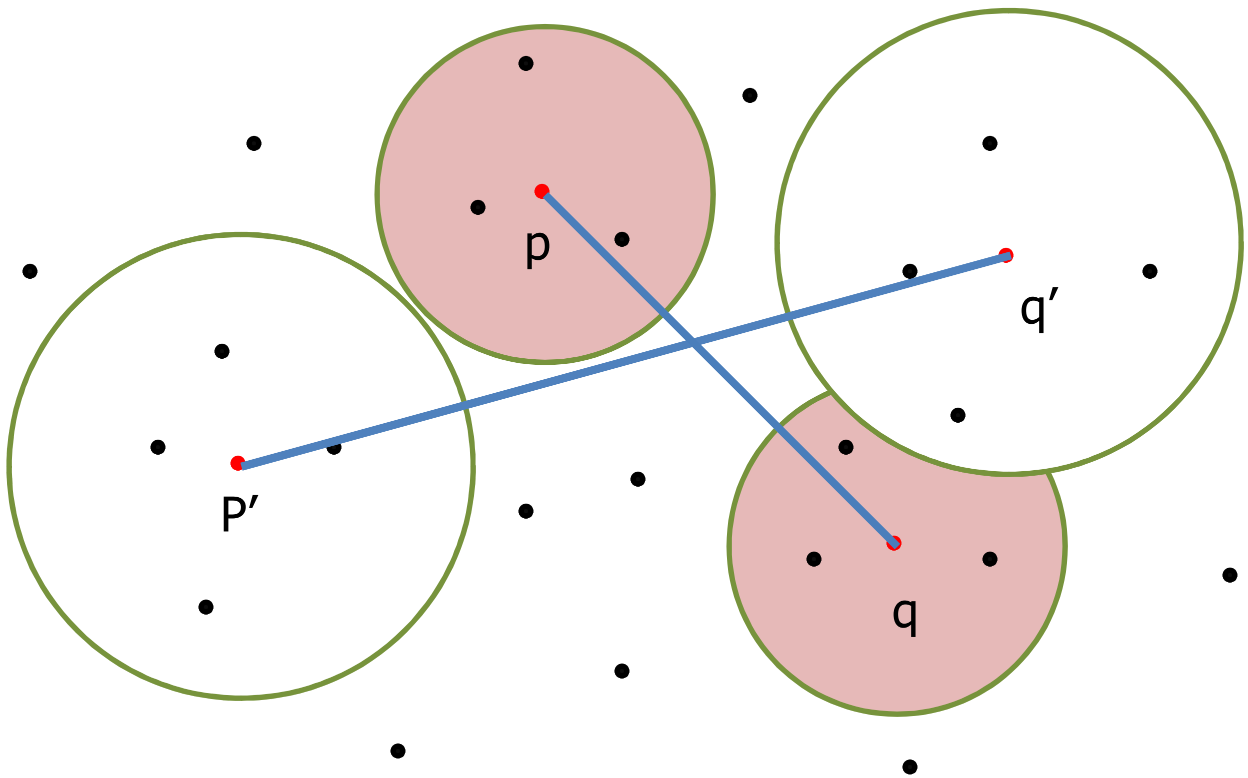

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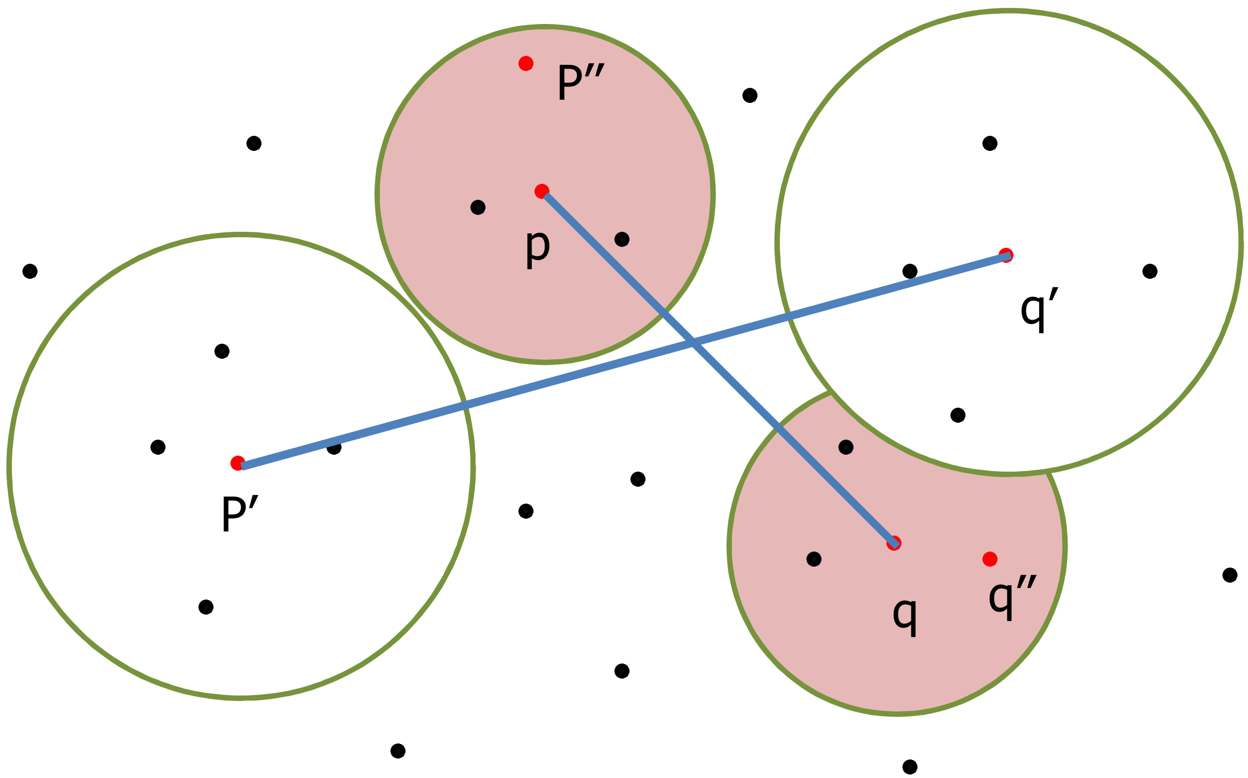

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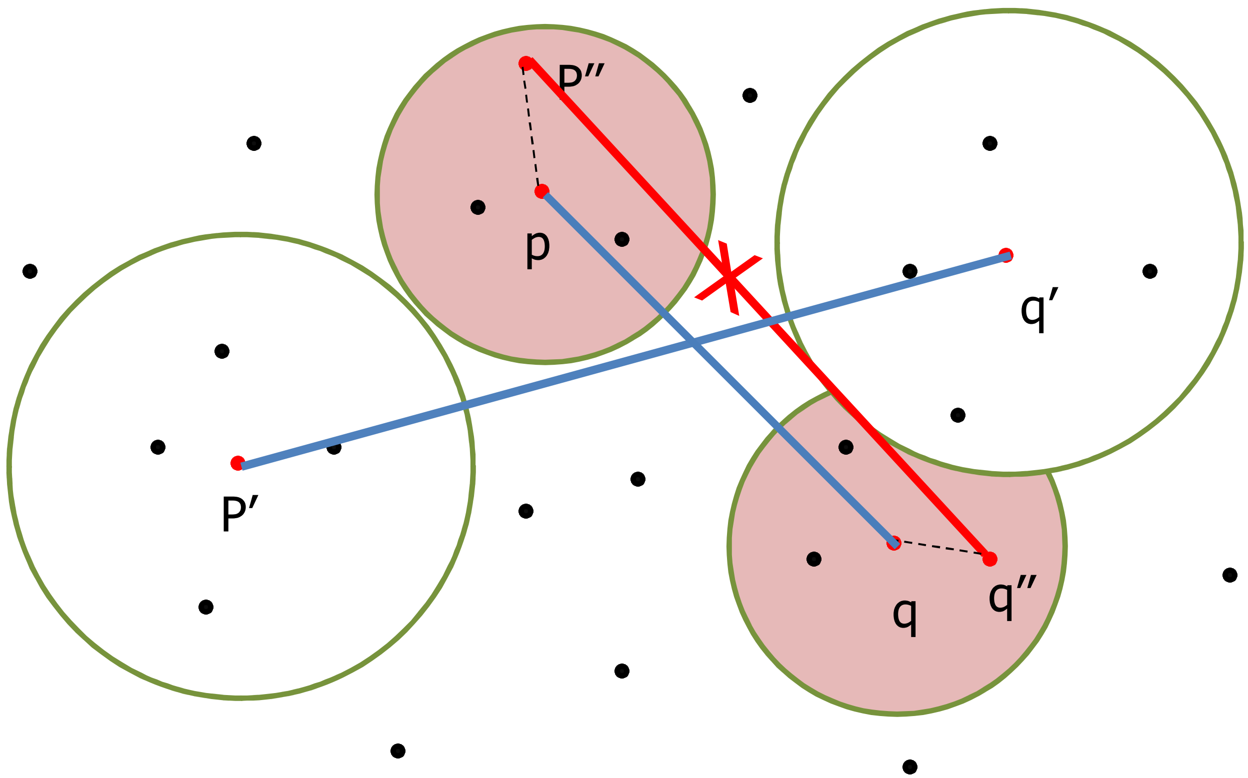


# Spanner Properties

- The graph constructed is a spanner
  - Stretch factor **$(s+1)/(s-1)$ ➔ $1+\varepsilon$**
  - Number of  edges: **$O(ns^d)$ ➔ $O(n/\varepsilon^d)$**
  - Maximal degree: $O(\lg\alpha\ s^d)$, $\alpha$ is aspect ratio.

    Average degree: $O(s^d)$
  - Total weight: $O(\lg\alpha\ |MST|\ s^{d+1})$
  - # hops for p,q : $2|pq|^{1/(1+\lg s)}$

# Next: Proof Sketch

- Connection to a greedy algorithm for well-separated pair decomposition (WSPD).
- Connection to the deformable spanner (GGN'04 SoCG).

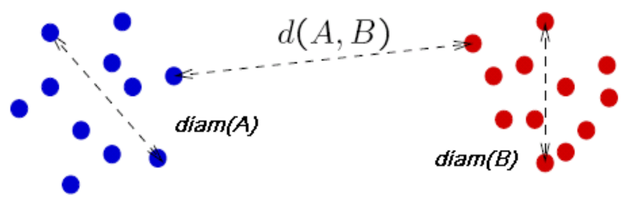
# Well-separated pair decomposition

- Point sets A, B are s-well-separated if

  $$d(A,B) \geq s \cdot \max(diam(A), diam(B))$$

  – Diameter $diam(A) = \max_{p,q \in A} d(p,q)$

  – Distance $d(A,B) = \min_{p \in A, q \in B} d(p,q)$



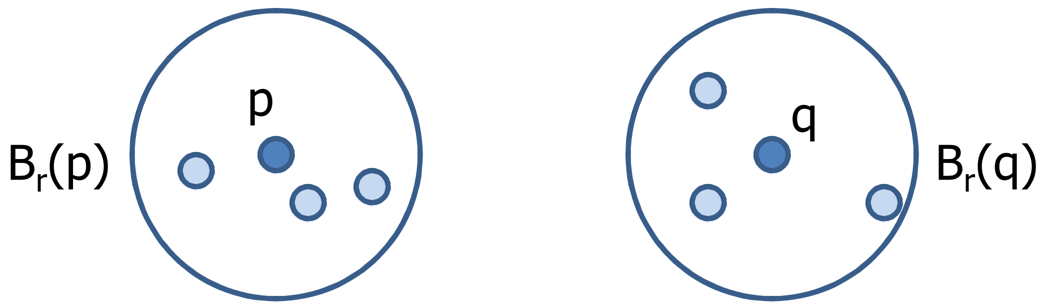

- For a set S of points, a set of pairs $P = \{(A_i, B_i)\}$ , is s-well-separated pair decomposition(WSPD) of S if

  – $(A_i, B_i)$ is s-well-separated.

  – For any two points $p \neq q \in S$ , there exists i such that $p \in A_i, q \in B_i$ .

# Greedy Algorithm for WSPD

- Choose an arbitrary "uncovered" pair (p, q).
- Include the "dumb-bell "pair of point set $B_r(p)$ and $B_r(q)$, with $r = |pq|/(2+2s)$




- Repeat the above steps until every pair of points is covered.
- Note: # pairs = # spanner edges = $\mathbf{O(n/\varepsilon^d)}$.

# Next: Proof Sketch

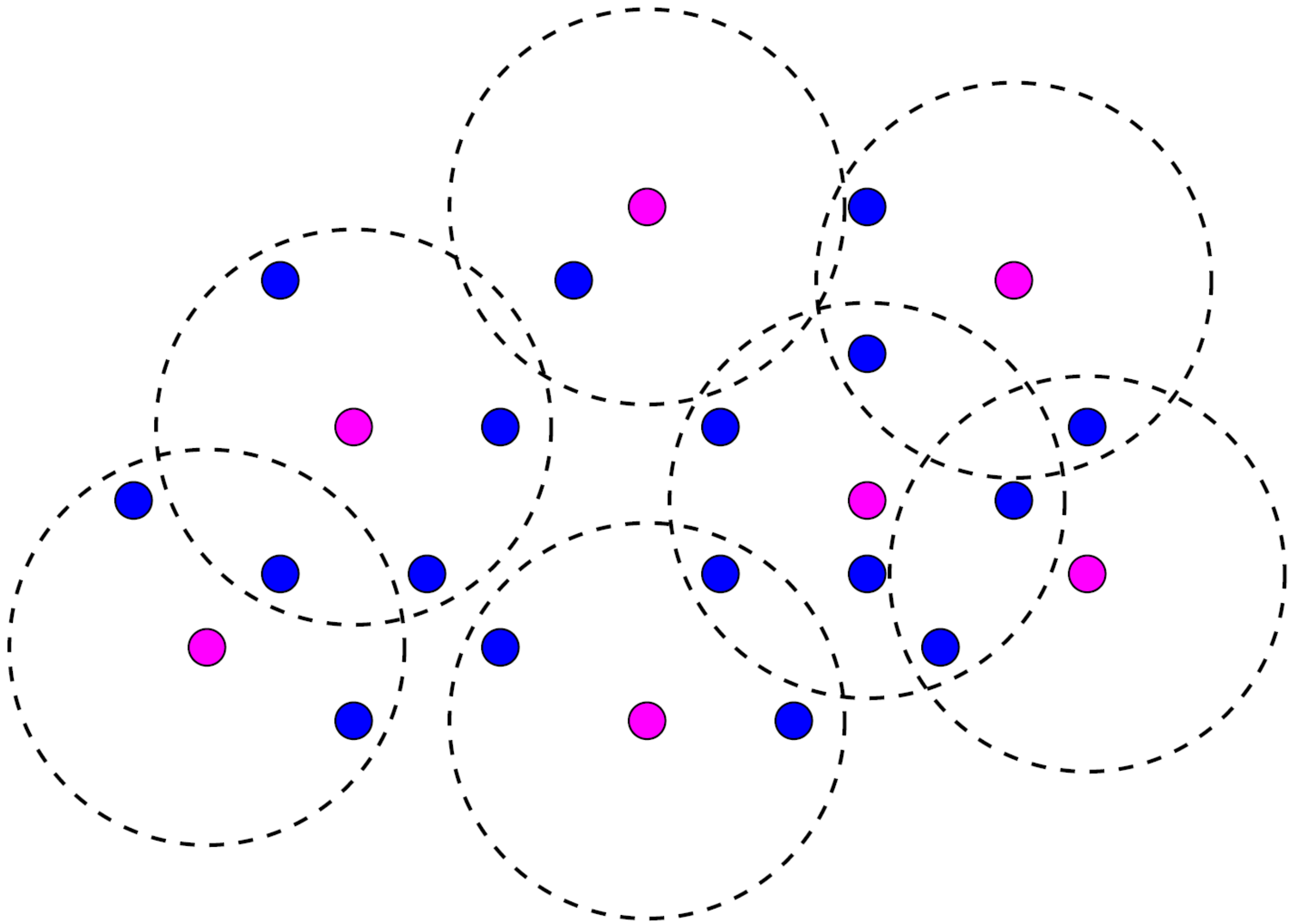
- Connection to a greedy algorithm for well-separated pair decomposition (WSPD).

- Connection to the deformable spanner (GGN'04 SoCG) ➔ proof of the linear spanner size.

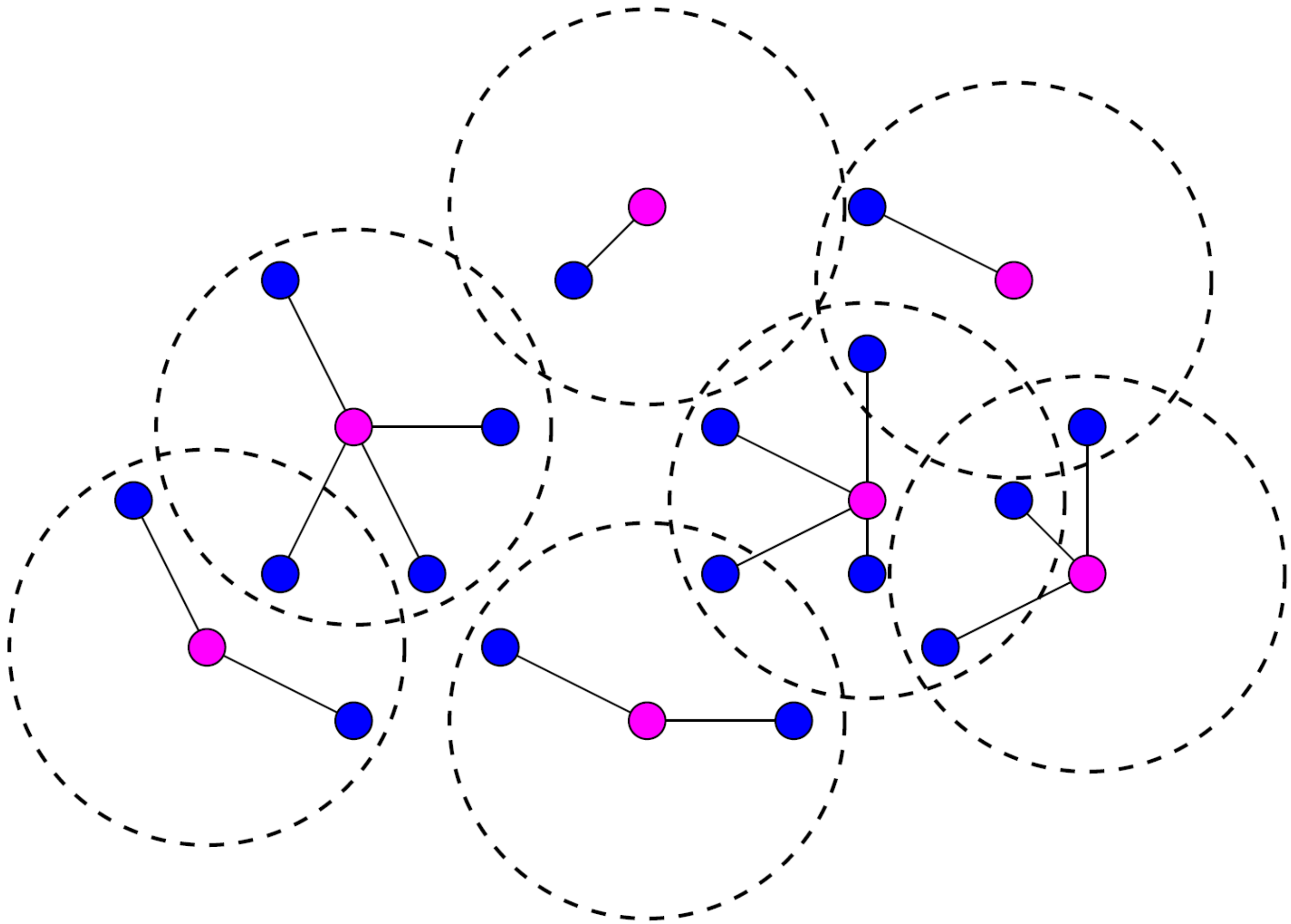
# Deformable spanner construction

- r-centers
  - Coverage: Every node is within distance r of at least one center
  - Separation: No two centers within distance r of each other
- (1+ ε)-Spanner
  - Build a hierarchical of centers
    - Centers at level **i** have radius $\mathbf{2^i}$.
  - At each level, add edges with length ≤ c $\mathbf{2^i}$.
    - c = 4+16/ε

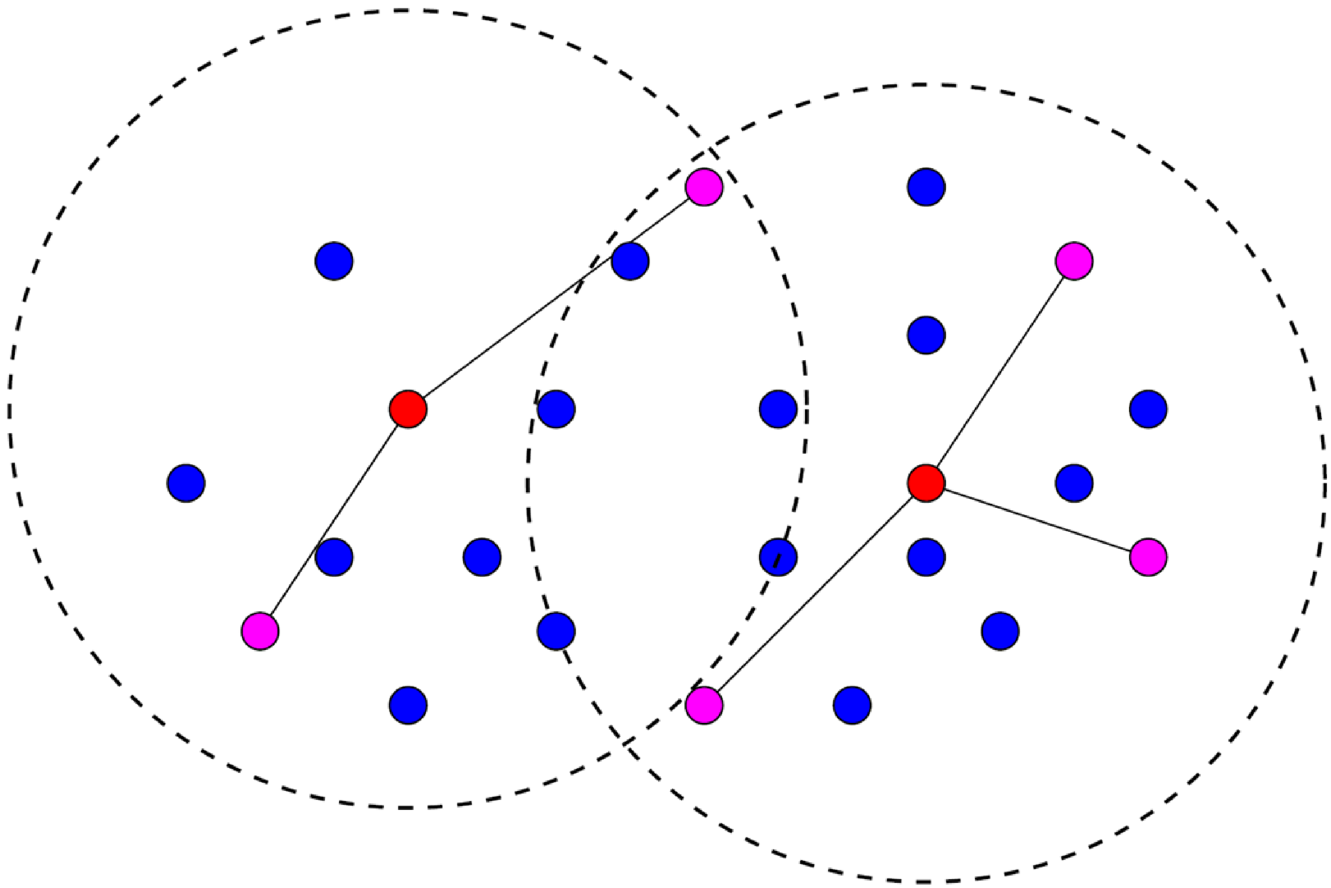
# $1^{st}$ level center



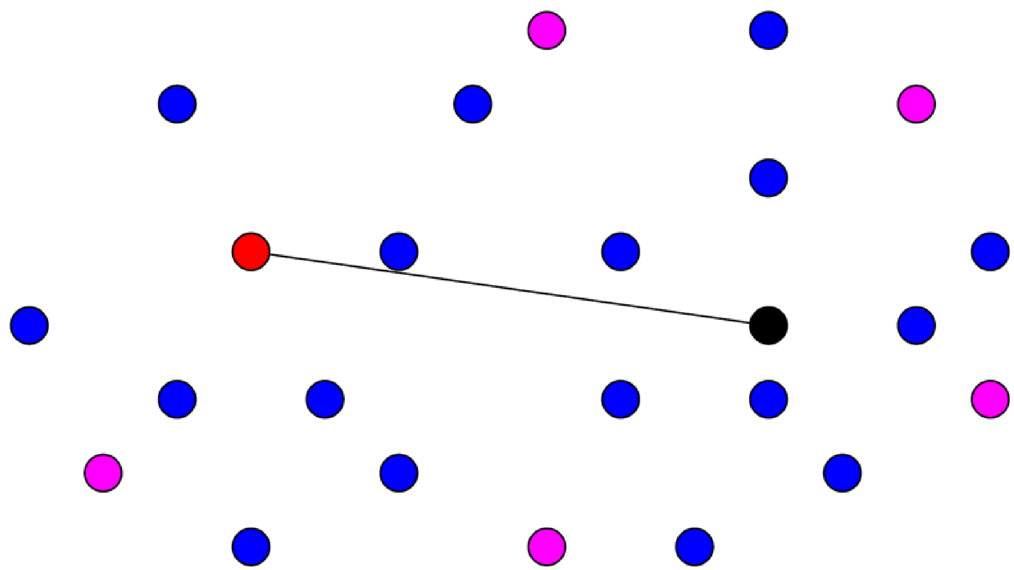
# Parent-child relationship



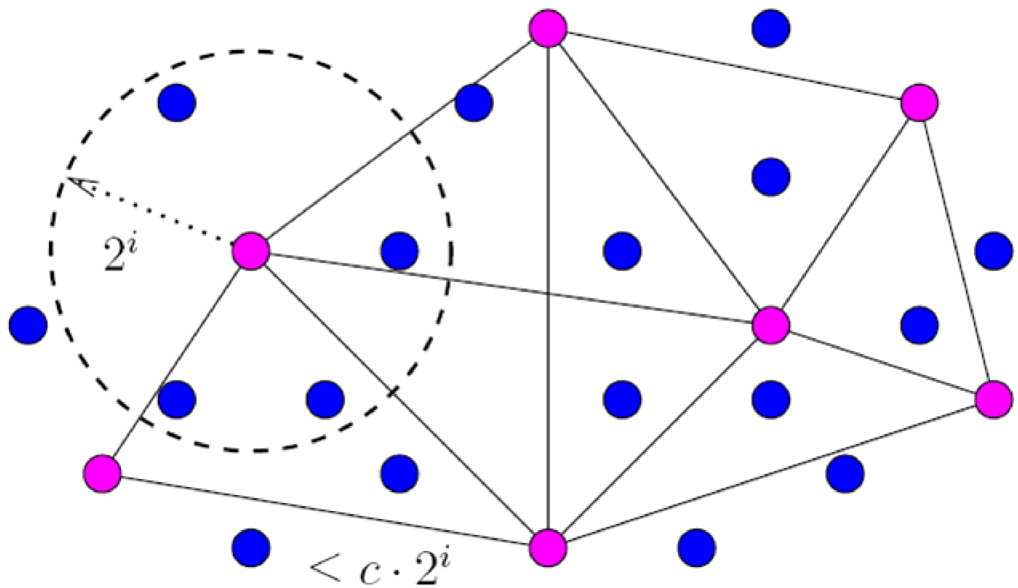
# $2^{nd}$ level center



# $3^{rd}$ level center



# 1st level spanner edges

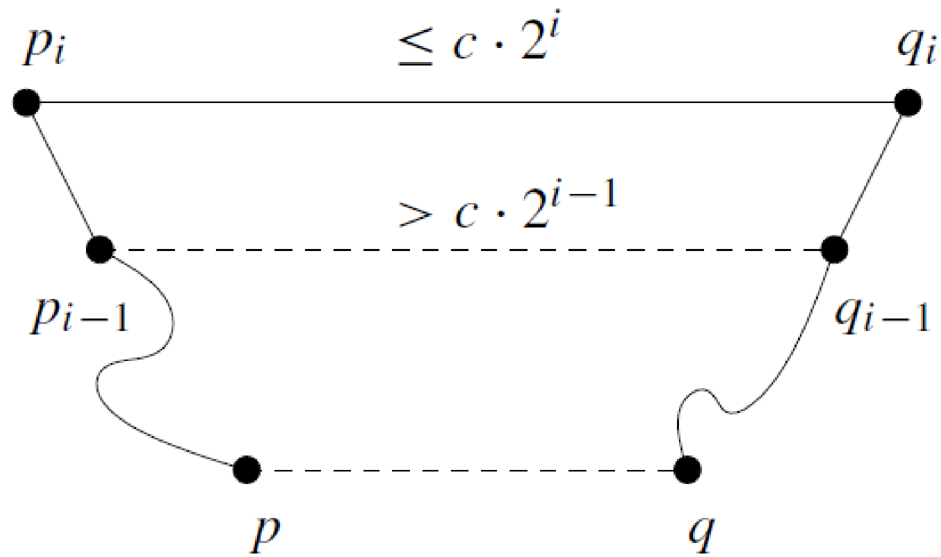


# Deformable Spanner

- A $(1+\varepsilon)$-spanner with **$O(n/\varepsilon^d)$** edges
- Maximal degree $O(\lg\alpha)$, $\alpha$ is aspect ratio.
- Total weight $O(\lg\alpha\ |MST|\ /\varepsilon^d)$

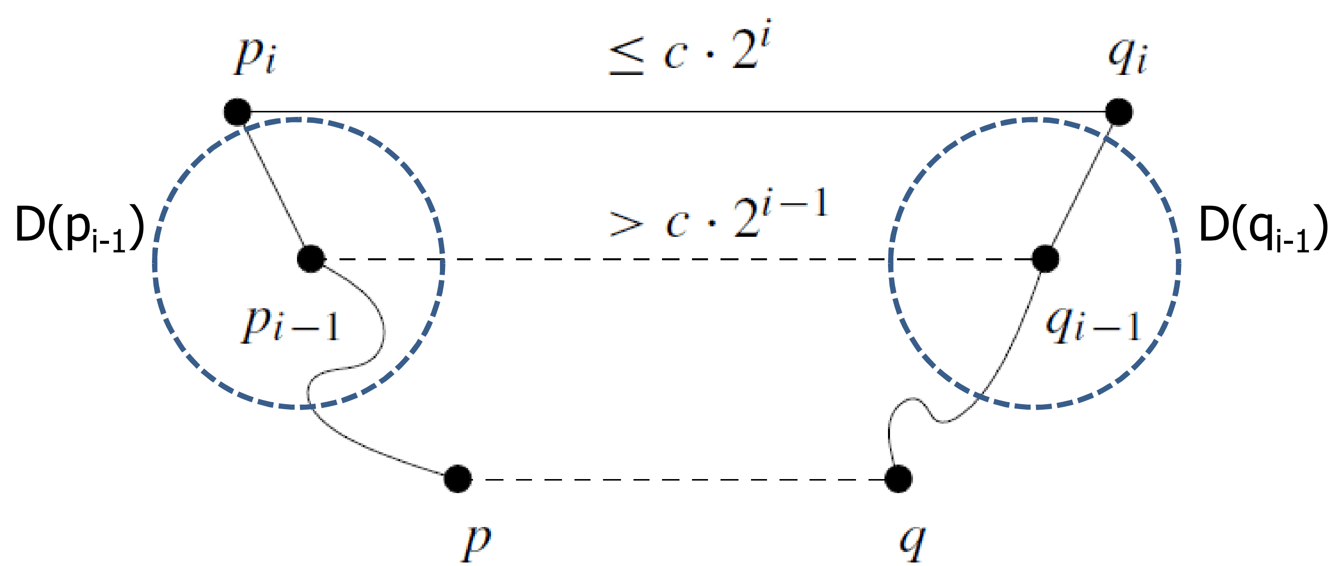
# Deformable spanner → WSPD

- In the deformable spanner, a pair of points (p,q) has level **i** ancestors ($p_i$ ,$q_i$) connected with an edge, but level **i-1** ancestors without an edge.



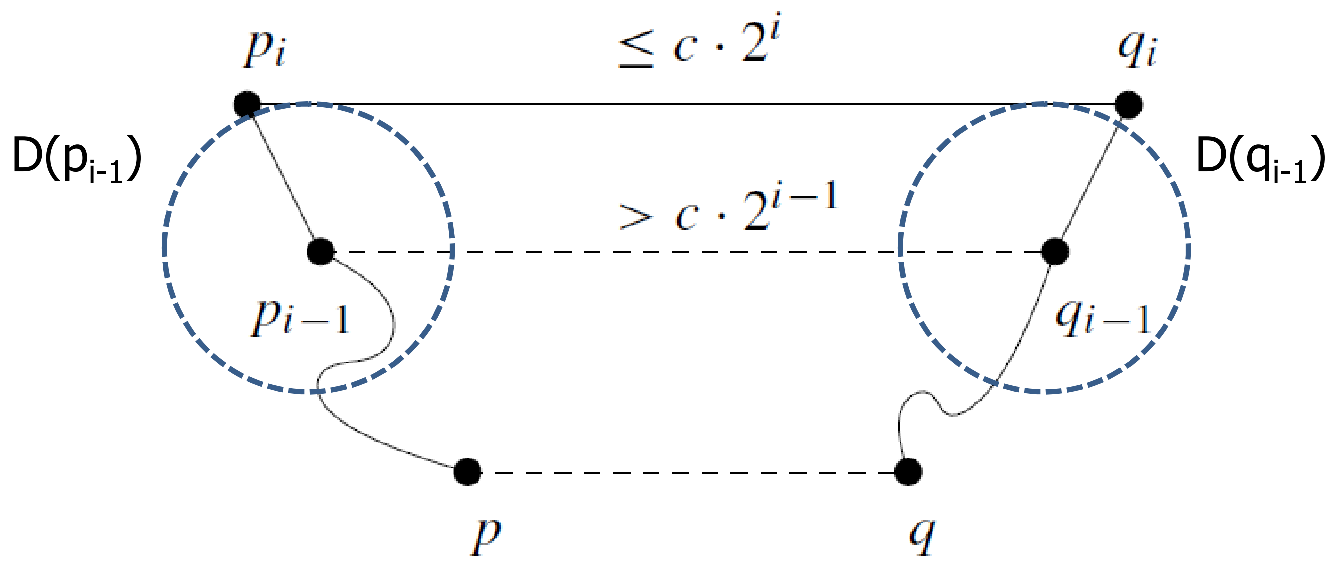
# Deformable spanner → WSPD

- For each edge ($p_i$ ,$q_i$), take the "cousin pair": the pair of sets of descendants (D($p_{i-1}$),D($q_{i-1}$)).
- All such pairs form a WSPD.



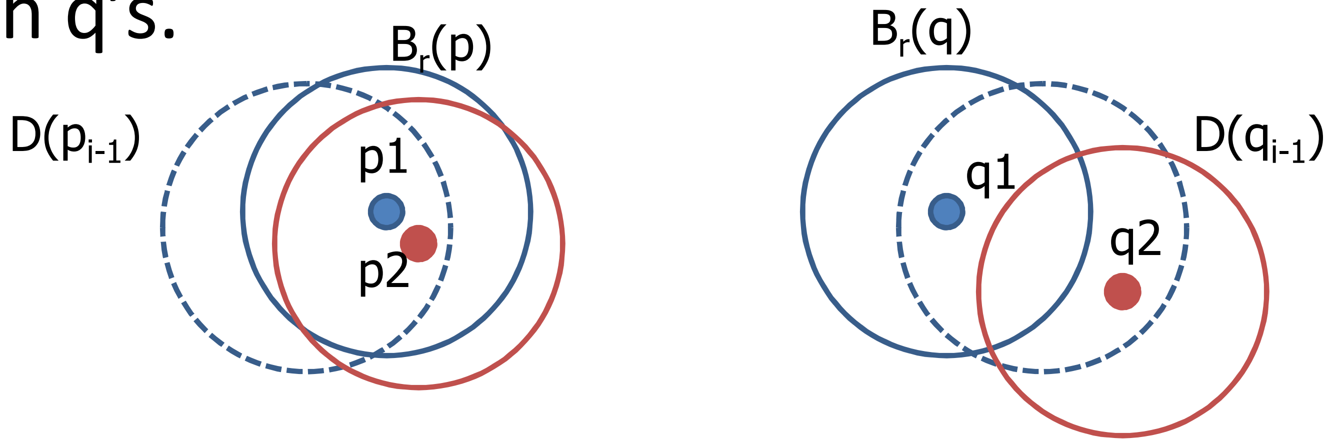
# Main Proof

- Our spanner: G, deformable spanner G'.
- Both imply WSPD, W, and W'.
- Main idea: A pair ($B_r(p)$, $B_r(q)$) maps to the "ancestor pair" ($D(p_{i-1})$,$D(q_{i-1})$).
- Argue only **O(1)** pairs of W map to one pair in W'.



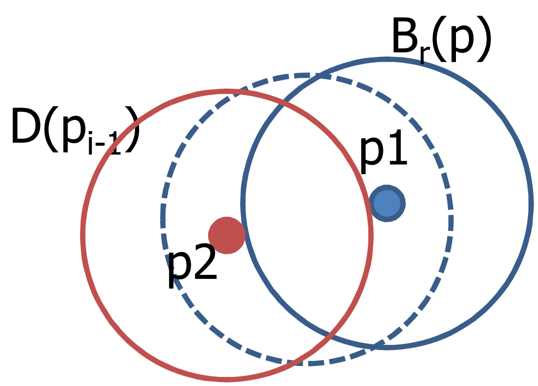
# Main Proof

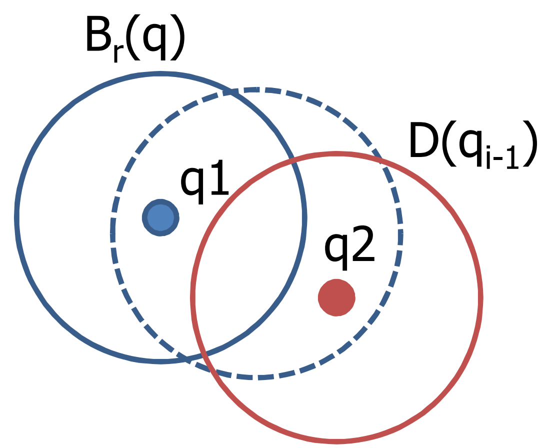
- Tool: packing argument.
- Consider two pairs (p1, q1) and (p2, q2) that map to the same pair in W'.
- **Case 1**: p1 and p2 are "close"; then q1 and q2 must be "far away". ➔ there are only O(1) such q's.



# Main Proof

- Tool: packing argument.
- Consider two pairs (p1, q1) and (p2, q2) that map to the same pair in W'.
- **Case 2**: p1 and p2 are "far away" ➔ there are only O(1) such p's.



# Application in P2P networks

- Proof only uses packing argument.
- Extension to metrics with constant doubling dimension
- De-centralized construction and maintenance of a spanner graph by distributed users.
  - Small message cost
  - Allow dynamic user insertion and deletion
  - Allow nearest neighbor query

# Future work

- Constant degree?
- Quality-varying spanner: In P2P network, punish “free-riders” and award diligent users.
  - A user get better stretch by constructing more edges.

Thanks!