Well-Separated Pair Decomposition for the Unit-disk Graph Metric and its Applications

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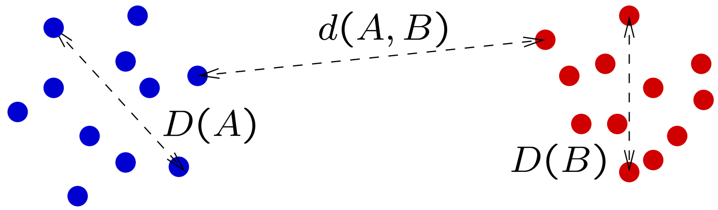
Geometric well-separated pair decomposition

Point sets A, B are c-well-separated if $d(A, B) \geq c \cdot \max(D(A), D(B))$.

- Diameter $D(A) = \max_{a,b \in A} d(a, b)$.
- Distance $d(A, B) = \min_{a \in A, b \in B} d(a, b)$.



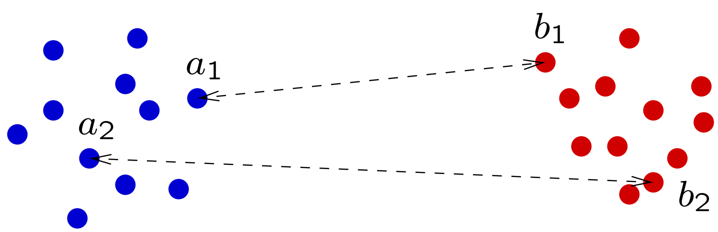
For a set S of points, a set of pairs $\mathcal{P} = \{(A_i, B_i)\}$, is a c-well-separated pair decomposition (WSPD) of S if

- (A_i, B_i) is c-well-separated.
- For any two points $a \neq b \in S$, there exists i such that $a \in A_i$, $b \in B_i$.

Geometric well-separated pair decomposition

Callahan and Kosaraju [1992] showed that a c-WSPD of $O(n)$ pairs for n points in $\mathbb{R}^d$ can be computed in $O(n \log n)$ time.

Observation: if (A, B) is c-well-separated, then for any $a_1, a_2 \in A$, $b_1, b_2 \in B$, $d(a_1, b_1) \leq (1 + 2/c)d(a_2, b_2)$.



WSPD approximates the $\Theta(n^2)$ pair-wise distances by $O(n)$ pairs.
$\Rightarrow$ Many problems involving all-pairs distances can be approximated by WSPD in almost linear time.

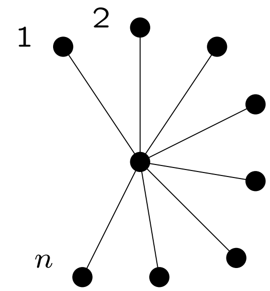
Apps: N-body problem, k-nearest neighbor, geometric spanners, etc.

Extend WSPD to other metrics?

Any metric admits WSPD of size $O(n^2)$: just take all the pairs of points.

Fact: general graph metric may not admit subquadratic WSPD. :-(

Example: c-WSPD of a star graph has size $\Omega(n^2)$, for $c > 1$.



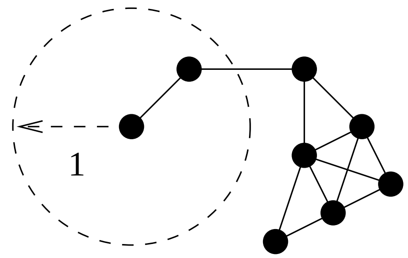
In this talk, we show that for n-points unit-disk graph metric, a WSPD of size $O(n \log n)$ can be found in $O(n \log n)$ time.

Overview

- Unit-disk graph (UDG) metric.
  - UDG is used to model wireless communication networks.
- Well-seprated pair decomposition for unit-disk graph metric:
  - Construction.
  - Bound of the size of the WSPD.
- Applications.
- Extensions to higher dimensions, unweighted unit-disk graph.
- Conclusion and future work.

Unit-disk graph metric

$I(S) = (S, E)$ is a weighted graph where an edge $e = (p, q) \in E$ if $d(p, q) \leq 1$. The weight of e is the Euclidean distance $d(p, q)$.



Unit-disk graph metric (S, π):

- $\pi(s_1, s_2)$ is the length of the shortest path in $I(S)$.
- Diameter $D(S')$ for $S' \subseteq S$ is $\max_{s_1, s_2 \in S'} \pi(s_1, s_2)$.
- Distance $\pi(S_1, S_2)$ for $S_1 \subseteq S$, $S_2 \subseteq S$ is $\min_{s_1 \in S_1, s_2 \in S_2} \pi(s_1, s_2)$.

WSPD for unit-disk graph

Theorem 1 *For unit-disk graph metric, a c-WSPD of size $O(c^4 n \log n)$ can be computed in time $O(c^4 n \log n)$.*

Proof. Define the density of the points to be the maximum number of points covered by any unit disk.

1. Nodes with constant bounded density.
   - Apply packing argument, which is similar with the collision detection algorithm of a necklace[Guibas02].
2. Nodes with arbitrary density.
   - Find a minimal cover (or clustering) with radius $O(1/c)$, cluster-heads have bounded density.

$\square$

WSPD construction

1. Construct a bounded degree spanning tree T.
2. Recursively decompose T balancedly by removing an edge once at a time.
3. Queue $Q \longleftarrow \{(S, S)\}$

   While $\{Q \neq \emptyset\}$ do

   Take (A, B) from Q, a, b are arbitrary points from A, B;

   if $d(a, b) \geq (c + 2) \cdot \max(|A| - 1, |B| - 1)$

   then output (A, B) to P;

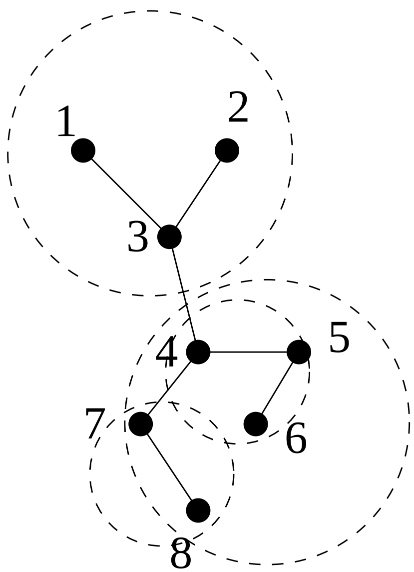
   else

   if $|A| = |B| = 1$, discard it, since they contain the same point.

   take the children of the larger one, say A_1, A_2 of A,

   put (A_1, B), (A_2, B) into Q.

Example



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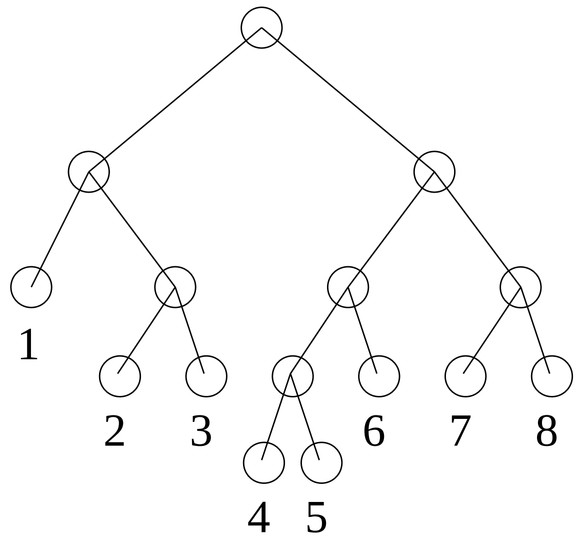


decomposition tree T'

({123}, {45678})

↓

({123}, {456}) Recurse

({123}, {78}) OK! output

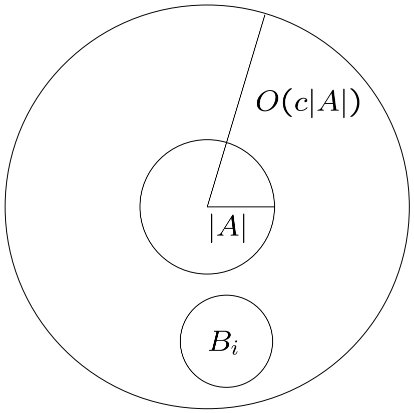
Bound the size of WSPD

Lemma 2 *P is a c-WSPD of S. Furthermore, each ordered pair of distinct points (p, q) is covered by exactly one pair in P.*

Proof. $d(a, b) \le \pi(a, b)$; $D(A) \le |A| - 1$. □

Lemma 3 *Each pair (A, B) that ever appears in the queue satisfies $1/\beta \le |A|/|B| \le \beta$, β is a constant.*

Proof. By induction. □



Lemma 4 *If $(A, B_i) \in P$, $i = 1, \cdots m(A)$, then $B_i \cap B_j = \emptyset$, and $m(A) = O(c^2|A|)$.*

Proof. Constant bounded density and packing argument. □

Bound the size of WSPD, cont.

Lemma 5 $|P| = O(c^2 n \log n)$.

Proof.

- The number of subsets with size in $[2^i, 2^{i+1}]$ is $O(n/2^i)$.
- They generate at most $O(c^2 2^i) \cdot O(n/2^i) = O(c^2 n)$ pairs.
- We sum up over $1 \leq i \leq \log n$.

$\square$

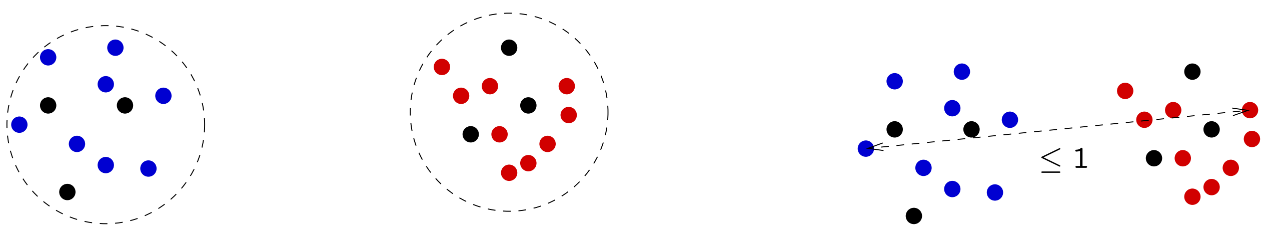
Also, running time: $O(c^2 n \log n)$.

We are all set with constant density case!

Arbitrary density

1. Find minimal cover, the clusterheads have bounded density $O(1/c^2)$.
2. Construct the WSPD on clusterheads.
3. Include for each clusterhead the other points it covers.

- For a far-away pair, including the other points still makes a well-separated pair.
- For a nearby pair, the distance between any points is smaller than 1, we apply the geometric WSPD.



(i) Far-away pairs (ii) Nearby pairs: geometric WSPD.

Estimate the distances

To use the WSPD, we need to compute $O(n \log n)$ pairs of distances.

- Preprocessing and distance queries.

1, Approximate distance query $\Rightarrow$ 2.42-approx. distance for any 2 points.

- Find a subgraph G, G is a planar $\frac{4\sqrt{3}}{9}\pi \approx 2.42$-spanner.
- Apply the distance oracle of [Thorup01] on the planar graph G.
  - Preprocessing time $O(n \log^3 n)$.
  - Each distance query takes $O(1)$ time.
- Running time: $O(n \log^3 n)$.

Estimate the distances, cont.

2, Exact distance query $\Rightarrow$ $(1+\varepsilon)$-approximate distance for any 2 points.

- Apply [Eppstein et.al.93] algorithm to find balanced separators.
- Extend the shortest distance algorithm by [Arikati et.al.96].
  - Preprocessing time $O(n^2/(\varepsilon r))$.
  - Each distance query takes $O(r/\varepsilon)$ time, $1 \leq r \leq \sqrt{n}$.
- Running time: $O(n\sqrt{n \log n}/\varepsilon^3)$.

Applications

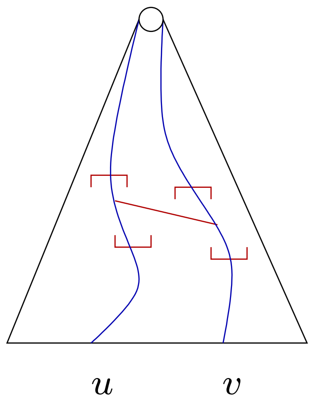
- $(1+\varepsilon)$-distance oracle with size $O(n \log n/\varepsilon^4)$ and $O(1)$ query time.
- For any $A, B \in S$, we get algorithms for:
  - Diameter of A.
  - Furthest neighbor of s with respect to A: $\max_{a \in A} \pi(s, a)$.
  - (Bichromatic) closest pair of A, B: $\min_{a \in A, b \in B} \pi(a, b)$.
  - Center with respect to A: $\min_{s \in S} \max_{a \in A} \pi(s, a)$.
  - Median with respect to A: $\min_{s \in S} \Sigma_{a \in A} \pi(s, a)$.
  - Stretch factor of a subgraph G_1: $\max_{s,t \in S} \frac{\pi_1(p,q)}{\pi(p,q)}$.

  with
  - $(1+\varepsilon)$-approximation in time $O(n\sqrt{n \log n}/\varepsilon^3)$.
  - 2.42-approximation in time $O(n \log^3 n)$.

Approximate distance oracle

Corollary 6 *For a unit-disk graph on n points and for any $\varepsilon > 0$, we can preprocess it into a data structure with $O(n \log n/\varepsilon^4)$ size so that for any query pair, a $(1+\varepsilon)$-approximate distance can be answered in $O(1)$ time.*

- Build c-WSPD. $c = 4/\varepsilon$.
- For each pair (A, B), compute the distances between (p, q), $p \in A$, $q \in B$.
- Query the 2.42-approximate distance for u, v.
- There is only $O(1)$ candidate pairs.



Note: Thorup's $(1 + \varepsilon)$-distance oracle for planar graph has size $O(n \log n)$ and query time $O(1/\varepsilon)$.

Example: bichromatic closest pair

Define the closest pair between $A, B \in S$ as $\min_{a \in A, b \in B} \pi(a, b)$.

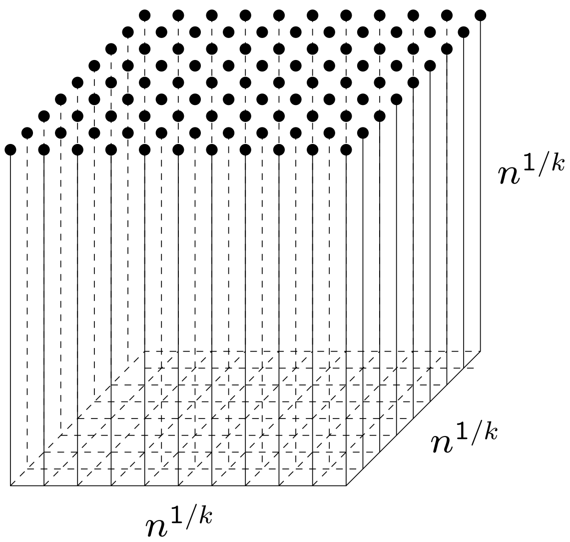
- Mark the node in T' that contains a point of A (B) accordingly.
- Mark a pair (A_i, B_i) of WSPD iff A_i and B_i are both marked.
- Find the smallest distance between all the marked pairs.

The others are similar.

Higher dimensions

For a unit-ball graph metric of n points in $\mathbb{R}^k$, $k \geq 3$, with constant bounded density and any constant $c \geq 1$, we can compute a c-WSPD with $O(n^{2-\frac{2}{k}})$ pairs in time $O(n^{2-\frac{2}{k}})$.

Note: This bound is tight in the worst case.



Unweighted unit-disk graph metric

Every edge in the unit-disk graph has weight 1.

- For points with constant-bounded density,
  - the bound on WSPD still works.
- For points with arbitrary density,
  - there may not be subquadratic WSPD, e.g., a complete graph.
  - for applications, we find the minimal cover first.

Conclusion and future work

- We extend WSPD to unit-disk graph metric and apply to many proximity problems.
- Open problems:
  - Size of the WSPD in the plane: $O(n \log n)$ and $\Omega(n)$. Close the gap?
  - Is there an almost linear-time $(1 + \varepsilon)$-approximation algorithm for $O(n \log n)$ shortest path queries?