Artificial Intelligence 303 (2022) 103630

Contents lists available at ScienceDirect

Artificial Intelligence

www.elsevier.com/locate/artint

Bayesian auctions with efficient queries ☆

Jing Chen [a], Bo Li [b,*], Yingkai Li [c], Pinyan Lu [d]

[a] Department of Computer Science, Stony Brook University, United States of America
[b] Department of Computing, The Hong Kong Polytechnic University, Hong Kong
[c] Department of Computer Science, Northwestern University, United States of America
[d] Institute for Theoretical Computer Science, Shanghai University of Finance and Economics, China

ARTICLE INFO

Article history:
Received 17 March 2021
Received in revised form 3 November 2021
Accepted 5 November 2021
Available online 10 November 2021

Keywords:
Mechanism design
The complexity of Bayesian mechanisms
Query complexity
Quantile queries
Value queries

ABSTRACT

Designing dominant-strategy incentive compatible (DSIC) mechanisms for a seller to generate (approximately) optimal revenue by selling items to players is a fundamental problem in Bayesian mechanism design. However, most existing studies assume that the seller knows the entire distribution from which the players' values are drawn. Unfortunately, this assumption may not hold in reality: for example, when the distributions have exponentially large supports or do not have succinct representations. In this work we consider, for the first time, the *query complexity* of Bayesian mechanisms. The seller only has limited oracle accesses to the players' distributions, via *quantile queries* and *value queries*. For single-item auctions, we design mechanisms with *logarithmic* number of value or quantile queries which achieve almost optimal revenue. We then prove logarithmic lower-bounds, i.e., logarithmic number of queries are necessary for any constant approximation DSIC mechanisms, even when randomized and adaptive queries are allowed. Thus our mechanisms are almost optimal regarding query complexity. Our lower-bounds can be extended to multi-item auctions with monotone subadditive valuations, and we complement this part with constant approximation mechanisms for unit-demand or additive valuation functions. Our results are robust even if the answers to the queries contain noises. Thus, in those settings the seller needs to access much less than the entire distribution to achieve approximately optimal revenue.

1. Introduction

An important problem in Bayesian mechanism design is to design auctions that (approximately) maximize the seller's expected revenue by selling items. More precisely, in a Bayesian multi-item auction a seller has m heterogeneous items to sell to n players. Each player has a private value for each item, which is independently drawn from some prior distribution. When the joint prior distribution is of *common knowledge* to both the seller and the players, optimal Bayesian incentive-

☆ We thank the editor and several anonymous reviewers for their helpful comments. Jing Chen is supported by NSF CAREER Award (No. 1553385). Bo Li is supported by The Hong Kong Polytechnic University (Grant No. P0034420). Pinyan Lu is supported by Science and Technology Innovation 2030 – "New Generation of Artificial Intelligence" Major Project No. (2018AAA0100903), NSFC grant 61922052 and 61932002, Innovation Program of Shanghai Municipal Education Commission, Program for Innovative Research Team of Shanghai University of Finance and Economics, and the Fundamental Research Funds for the Central Universities. Part of this work was done when the first three authors were visiting Shanghai University of Finance and Economics. A brief announcement of this paper has appeared on 45th International Colloquium on Automata, Languages, and Programming.

* Corresponding author.
E-mail addresses: jingchen@cs.stonybrook.edu (J. Chen), comp-bo.li@polyu.edu.hk (B. Li), yingkai.li@u.northwestern.edu (Y. Li), lu.pinyan@mail.shufe.edu.cn (P. Lu).

https://doi.org/10.1016/j.artint.2021.103630

Table 1
Our main results. Here $h(\cdot) < 1$ is the tail function in the small-tail assumptions. For single-item auctions, the revenue is a $(1+\epsilon)$-approximation to the optimal BIC revenue, with ϵ sufficiently small. For multi-item auctions with unit-demand or additive valuation functions, the revenue is a c-approximation for some constant c. The upper bounds in the table holds for queries with small noise in the response while the lower bounds holds even for queries without noise. Note that the result of regular distributions doesn't require the distribution to be bounded.

	Query complexity	Distributions		
		Bounded in $[1, H]$		Unbounded & small tail
Auctions	Single-Item	$\Theta(n\epsilon^{-1}\log H)$		$O(-n\epsilon^{-1}\log h(\frac{2\epsilon}{3(1+\epsilon)}))$
	Unit-Demand	$\forall c > 1$: $\Omega(\frac{mn\log H}{\log c})$	$\forall c > 24$: $O(\frac{mn\log H}{\log(c/24)})$	$\forall c > 24$: $O(-\frac{mn\log h(\frac{2c-48}{3c})}{\log(c/24)})$
	Additive	$\forall c > 1$: $\Omega(\frac{mn\log H}{\log c})$	$\forall c > 8$: $O(\frac{mn\log H}{\log(c/8)})$	$\forall c > 8$: $O(-\frac{m^2 n\log h(\frac{c-8}{10c})}{\log(c/8)})$
	Single-Item	Regular Distributions: $\Omega(n\epsilon^{-1})$, $O(n\epsilon^{-1}\log\frac{n}{\epsilon})$		

compatible (BIC) mechanisms have been discovered for various auction settings [39,19,11], where all players reporting their true values forms a Bayesian Nash equilibrium. When there is no common prior but the seller knows the distribution, many (approximately) optimal dominant-strategy incentive-compatible (DSIC) Bayesian mechanisms have been designed [39,41,15,12], where it is each player's *dominant strategy* to report his true values.

However, the *complexity* for the seller to carry out such mechanisms is largely unconsidered in the literature. Most existing Bayesian mechanisms require that the seller has full access to the prior distribution and is able to carry out all required optimizations based on the distribution, so as to compute the allocation and the prices. Unfortunately the seller may not be so knowledgeable or powerful in real-world scenarios. If the supports of the distributions are exponentially large (in m and n), or if the distributions are continuous and do not have succinct representations, it is hard for the seller to write out "each single bit" of the distributions or precisely carry out arbitrary optimization tasks based on them. Even with a single player and a single item, when the value distribution is irregular, computing the optimal price in time that is much smaller than the size of the support is not an easy task. Thus, a natural and important question to ask is *how much the seller should know about the distributions in order to obtain approximately optimal revenue.*

In this work we consider, for the first time, the *query complexity* of Bayesian mechanisms. In particular, the seller can only access the distributions by making oracle queries. Two types of queries are allowed, *quantile queries* and *value queries*. That is, the seller queries the oracle with specific quantiles (respectively, values), and the oracle returns the corresponding values (respectively, quantiles) in the underlying distributions. These two types of queries happen a lot in market study. Indeed, the seller may wish to know what is the price he should set so that half of the consumers would purchase his product; or if he sets the price to be 200 dollars, how many consumers would buy it. Another important scenario where such queries naturally come up is in databases. Indeed, although the seller may not know the distribution, some powerful institutes, say the Office for National Statistics, may have such information figured out and stored in its database. As in most database applications, it may be neither necessary nor feasible for the seller to download the whole distribution to his local machines. Rather, he would like to access the distribution via queries to the database. Moreover, the commercial data providers may charge the mechanism designer a fixed monetary payment for each query about the statistics of the distribution. To maximize the revenue of the designer, it is necessary to understand the tight bounds on the number of queries that are sufficient for good revenue guarantees.

Another concern in practice is that the queries to the valuation distributions are not precise. One reason for the occurrence of such error is that the data provider only has estimates for the statistics of the valuation distributions, and those estimates are only approximately correct with high probability. Another reason is that the data provider may even try to obfuscate the data due to privacy issues. For example, when the data provider adopts techniques such as differential privacy [25], the data provider will add a noise to the returned value of each query. In this paper, we show that our results are robust with the presence of imprecise queries.

In this work we mainly focus on *non-adaptive* queries, where later queries cannot be based on the answers of previous queries. Thus the seller makes all oracle queries simultaneously, before the auction starts. This also happens in both database and market study scenarios, and in Section 4, we will show that the performance of adaptive queries cannot be improved by more than a logarithmic factor.

1.1. Main results

We would like to understand both lower- and upper-bounds for the query complexity of approximately optimal Bayesian auctions. In this work, we will first consider single-item auctions and then extend our results to multi-item settings when the players' valuations are either unit-demand or additive. When the distributions are bounded within $[1, H]$ and the queries are precise, our results are summarized in Table 1. We also show that our query complexity results extend for arbitrary unbounded distributions that satisfy *small-tail assumptions*, with formal definitions provided in Section 3.2.1. These results are summarized in Table 1 as well. Similar small-tail assumptions are widely adopted in sampling mechanisms [42,22], to deal with irregular distributions with unbounded supports. Finally, in Section 3.3, we show that the revenue of the query mechanisms degrade proportionally to the amount of error contained in the queries to the valuation distributions.

Table 2
Sample complexity in the literature. For multi-item auctions with unit-demand or additive valuation functions, the revenue has an extra ϵ additive loss. The results for single-item auctions are by [22], for unit-demand auctions are by [38], and for additive auctions are by [9], respectively.

Auctions	Single-item (regular)	Single-item (bounded in $[1, H]$)	Unit-demand (bounded in $[1, H]$)	Additive (bounded in $[1, H]$)
Sample complexity	$\tilde{O}(n\epsilon^{-4})$	$\tilde{O}(nH\epsilon^{-3})$	$\tilde{O}(nm^2H^2\epsilon^{-2})$	$\tilde{O}(nm^2H^2\epsilon^{-2})$
Approximations	$1+\epsilon$	$1+\epsilon$	27	32

Following the convention of the literature, the mechanism designer implements DSIC mechanisms while the benchmark is the optimal BIC revenue. It is known that the gap between the optimal DSIC mechanism and the optimal BIC mechanism for multi-item setting is bounded away from 1 [45], and the best known approximation ratio for DSIC mechanisms to the optimal BIC revenue in unit-demand and additive auctions are 24 and 8 respectively in the literature [12]. The main focus of this paper is not the tightness of approximation ratios for those mechanisms, but the amount of information required for the mechanism designer to achieve such approximations.

Also note that our lower- and upper-bounds on query complexity are *tight* for bounded distributions. As will become clear in Section 4, our lower-bounds allow the seller to make both value and quantile queries, and apply to any multi-player multi-item auctions where each player's valuation function is *succinct sub-additive*: formal definitions in Section 4. The lower-bounds also allow randomized adaptive queries and randomized mechanisms.[1]

For the upper-bounds, all our query schemes are deterministic, non-adaptive, and only make one type of queries: value queries for bounded distributions and quantile queries for unbounded distributions with small-tail assumptions. We show that our schemes, despite of being very efficient, only loses a small fraction of revenue compared with when the seller has full access to the distributions.

1.2. Related work

Sampling mechanisms A closely related area to our work is sampling mechanisms [18,22,5,29,34,1]. It assumes that the seller does not know the distribution D but observes random samples from D before the auction begins. There are two branches of researches for sampling mechanisms. One branch focuses on designing mechanisms that minimize the approximation ratio given only one sample [35,1] or two samples [5,21]. The other one is referred as sample complexity, which measures how many samples the seller needs so as to obtain a good approximation to the optimal Bayesian revenue [9,29,8]. Most of the previous results focus on analyzing the sample complexity of DSIC mechanisms, which is similar to what we adopt in this paper. There are two exceptions. In Gonczarowski and Nisan [27], the authors provide upper bounds on the sample complexity for $(1+\epsilon)$-approximation to the optimal BIC revenue. However, the solution concept adopted there is ϵ-BIC, and the revenue guarantee under Bayesian Nash equilibrium for the mechanism proposed there is unclear. Hartline and Taggart [34] resolve the issue for the single-item setting, and provide upper bounds on the sample complexity of non-truthful mechanisms for $(1+\epsilon)$-approximation to the optimal BIC revenue, under the solution concept of Bayesian Nash equilibrium. The best-known sample complexity results are summarized in Table 2.

Oracle queries can be seen as *targeted samples*, where the seller actively asks the information he needs rather than passively learns about it from random samples. As such, it is intuitive that queries are more efficient than samples, but it is a priori unclear how efficient queries can be. Our results answer this question quantitatively and show that query complexity can be exponentially smaller than sample complexity: the former is *logarithmic* in the "size" of the distributions, while the latter is polynomial.[2] Finally, the design of query mechanisms facilitates the design of sampling mechanisms. If the seller observes enough samples from D, then he can mimic quantile queries and apply query mechanisms: see Section 7 for more details.

Parametric auctions Parametric mechanisms [3,14,17] assume the seller only knows some specific parameters about the distributions, such as the mean, the median (or a single quantile), the variance or higher moments. Note that using quantile or value queries, one can get the exact value of the median and the approximate value of the mean, and then apply parametric mechanisms. Existing parametric mechanisms only consider single-parameter auctions. Since our mechanisms make non-adaptive oracle queries, our results imply parametric mechanisms in multi-parameter settings with general distributions, where the "parameters" are the oracle's answers to our query schemes. Our lower-bounds also imply that knowing only the median is not enough to achieve the same approximation ratios as we do.

Distributions within bounded distance There are several papers in the literature that consider the model where the seller is given a distribution that is within a small distance to the true prior distribution. For single-item single-buyer settings, Bergemann and Schlag [7] characterize the optimal robust monopoly pricing under the Prohorov distance. Li et al. [37]

1 As shown in Theorem 3 and 4, introducing randomization in the mechanisms does not affect the bounds, while when the adaptive queries are allowed, there is an additional multiplicative factor of $\frac{1}{\log(m \log H)}$ in the lower bound.

2 For example, in the single-item auction, the sample complexity is $\tilde{O}(n^2H\epsilon^{-2})$ when the valuations are bounded in $[1, H]$.

consider the same problem with the earth-mover distance and extends the characterization to the multi-buyer setting. Recently, [9] study the multi-item auctions under the Kolmogorov distance. The distinction between our model and those papers is that the learnt distributions from our query schemes may be far from the true prior in terms of those specified distances (e.g., Prohorov, Kolmogorov or earth-mover), and thus their mechanisms do not apply. On the other hand, although a distribution close to the true prior may be learnt via sufficiently many oracle queries given Prohorov distance, our results imply that the query complexity of this approach will not be better than ours.

Menu complexity The complexity of auctions is an important topic in the literature, and other complexity measures such as menu complexity have been considered. Following the taxation principle [30,28,31], defines the *menu complexity* of truthful auctions. For a single additive buyer, [20] show the optimal Bayesian auction for revenue can have an infinite menu size or a continuum of menu entries, and [6] bound the menu complexity for approximating the optimal revenue. Recently, [24] considers the taxation, communication, query and menu complexities of truthful combinatorial auctions, and shows important connections among them. The queries considered there are totally different from ours: we are concerned with the complexity of accessing the players' value distributions in Bayesian settings, while [24] is concerned with the complexity of accessing the players' valuation functions in non-Bayesian settings.

2. Preliminaries

2.1. Bayesian auctions

In a multi-item auction there are m items, denoted by $M = \{1, \dots, m\}$, and n players, denoted by $N = \{1, \dots, n\}$. Each player $i \in N$ has a non-negative value for each item $j \in M$, v_{ij}, which is independently drawn from distribution D_{ij}. Player i's *true valuation* is $(v_{ij})_{j\in[m]}$. To simplify the notations, we may write v_i for $(v_{ij})_{j\in[m]}$ and v for $(v_i)_{i\in[n]}$. Letting $D_i = \times_{j\in M} D_{ij}$ and $D = \times_{i\in N} D_i$, we use $\mathcal{I} = (N, M, D)$ to denote the corresponding Bayesian auction instance. We will consider several classes of widely studied auctions. A *single-item* auction has $m = 1$. When $m > 1$, a player i being *unit-demand* means his value for a subset S of items is $\max_{j\in S} v_{ij}$, and a player i being *additive* means his value for S is $\sum_{j\in S} v_{ij}$. When all players are unit-demand (respectively, additive), we call such an auction a *unit-demand auction* (respectively, an *additive auction*) for short.

A mechanism $\mathcal{M}$ maps a reported value profile v from the players to a/n (random) allocation of items, $x = (x_i(v))_{i\in N}$, and payments to charge the players, $p = (p_i(v))_{i\in N}$. When v is clear from the context, we simply denote by $x = (x_i)_{i\in N}$ and $p = (p_i)_{i\in N}$. For single-item auctions, $x_i \in [0, 1]$ and for multi-item auctions, $x_i \in [0, 1]^m$. A mechanism is called Bayesian Incentive Compatible (BIC) if it is every player's optimal strategy to report her true value, given all other players report truthfully. A mechanism is called Dominant Strategy Incentive Compatible (DSIC) if it is every player's optimal strategy to report her true value no matter what values are reported by the other players. Given any instance $\mathcal{I}$, let $\mathrm{Rev}(\mathcal{M}(\mathcal{I}))$ be the expected revenue generated by $\mathcal{M}$ and $\mathrm{OPT}(\mathcal{I})$ be the optimal BIC revenue of $\mathcal{I}$, i.e., the maximum expected revenue generated by BIC mechanisms. When $\mathcal{I}$ is clear from the context, we write OPT for short. A mechanism achieves c-approximation if for any instance $\mathcal{I}$, $\mathrm{Rev}(\mathcal{M}(\mathcal{I})) \ge \frac{\mathrm{OPT}(\mathcal{I})}{c}$.

2.2. Query complexity

In this work, we only allow the seller to access the prior distributions via two types of oracle queries: *value queries* and *quantile queries*. Given a distribution D over real numbers, in a value query, the seller sends a value $v \in \mathbb{R}$ and the oracle returns the corresponding quantile $q(v) \triangleq \Pr_{x\sim D}[x \ge v]$. In a quantile query, the seller sends a quantile $q \in [0, 1]$ and the oracle returns the corresponding value $v(q)$ such that $\Pr_{x\sim D}[x \ge v(q)] = q$. With *non-adaptive* queries, the seller first sends all his queries to the oracle, gets the answers back, and then runs the auction. The *query complexity* is the number of queries made by the seller.

Note that the answer to a value query is unique. The quantile queries are a bit tricky, as for discrete distributions there may be multiple values corresponding to the same quantile q, or there may be none. When there are multiple values, to resolve the ambiguity, let the output of the oracle be the largest one: that is, $v(q) = \arg\sup_z\{\Pr_{x\sim D}[x \ge z] \ge q\}$.[3] Note that for any discrete distribution D and quantile query $q > 0$, $v(q)$ is always in the support of D. Moreover, when $q = 0$, $v(q)$ may be $+\infty$.

Noisy queries In this paper, we also consider the model where the answers to the queries contain errors, i.e., noisy value queries and noisy quantile queries.[4]

Definition 1. For any distribution D and $\eta > 0$, a value query has η-noise if for any value $v \in \mathbb{R}$, the returned quantile is

$$q \in \left[\frac{1}{1+\eta} \cdot \Pr_{x\sim D}[x \ge v],\ (1+\eta) \cdot \Pr_{x\sim D}[x \ge v]\right].$$

[3] The tie breaking rule here is chosen to simplify the exposition. All of our results extend to arbitrary tie breaking rules.

[4] In Definition 1, we define the noise as multiplicative errors. This choice is made since we focus on multiplicative approximations in our paper. Similar results can be obtained for additive error, which will not be elaborated in this paper.

Algorithm 1 The value-query algorithm $\mathcal{A}_V$.

Input: The value bound H and the precision factor δ.

Let $k = \lceil \log_{1+\delta} H \rceil$ and define the *value vector* as $v = (v_0, v_1, \ldots, v_{k-1}, v_k) = (1, (1+\delta), (1+\delta)^2, \ldots, (1+\delta)^{k-1}, H)$.

Query the oracle for D with v, and receive a non-increasing quantile vector $q = (q(v_0), \cdots, q(v_k)) = (q_l)_{l \in \{0,\ldots,k\}}$. Note $q_0 = 1$.

Construct a discrete distribution D' as follows: $D'(v_l) = q_l - q_{l+1}$ for any $l \in \{0, \ldots, k\}$, where $q_{k+1} \triangleq 0$.

Output: Distribution D'.

Mechanism 2 Efficient value Myerson mechanism $\mathcal{M}_{EVM}$.

Given the value bound H and a constant $\epsilon > 0$, run the value-query algorithm $\mathcal{A}_V$ with H and $\delta = \epsilon$ for each player i's distribution D_i. Denote by D'_i the returned distribution. Let $D' = \times_{i \in N} D'_i$.

Run $\mathcal{M}_{MRS}$ with D' and the players' reported values, $b = (b_i)_{i \in N}$, to get allocation $x = (x_i)_{i \in N}$ and price profile $p = (p_i)_{i \in N}$ as the outcome.

Similarly, a quantile query has η-noise if for any quantile $q \in [0, 1]$, the returned value v satisfies

$$v \in \left[\frac{1}{1+\eta} \cdot v', (1+\eta) \cdot v' \right] \text{ and } v' = \arg\max_z \{ \Pr_{x \sim D} [x \geq z] \geq q \}.$$

3. Efficient query mechanisms for single-item auctions

In this section, we first focus on designing efficient query mechanisms for single-item auctions when the queries are precise. The generalizations to η-noise queries are discussed in Section 3.3. We will consider both settings where all distributions are bounded or unbounded but satisfy small tail assumptions (formally defined in Section 3.2.1). Our results in this section rely on Myerson's characterization of the optimal mechanism for single-item auctions [39]. For any valuation distribution with cumulative distribution function F and density f, the *virtual value* is defined as $\phi_F(v) = v - \frac{1-F(v)}{f(v)}$.

Lemma 1 *([39]). For any BIC mechanism $\mathcal{M}$ with allocation rule x, any instance $\mathcal{I} = (N, M, D)$, the expected revenue equals the expected virtual welfare, i.e.,*

$$\mathrm{Rev}(\mathcal{M}(\mathcal{I})) = \mathbb{E}_{v \sim D} \sum_i [\phi_F(v_i) \cdot x_i(v)].$$

Thus, in Myerson's mechanism, we first map all players' values to virtual values and then run the second price auction with reserve price 0 on the virtual values, which achieves the optimal revenue.

3.1. Bounded distributions and value-query mechanisms

We first consider the setting when all distributions are bounded within $[1, H]$. We show that it is sufficient to use only value queries, and we define a universal query scheme $\mathcal{A}_V$, which will be used as a black-box in our mechanisms. The seller uses algorithm $\mathcal{A}_V$ to learn a distribution $D' = \times_{i \in N, j \in M} D'_{ij}$ that approximates the prior distribution D and is first order stochastically dominated by D (i.e., $\Pr_{x \sim D_{ij}}[x \geq v] \geq \Pr_{x \sim D'_{ij}}[x \geq v]$ for any i, j and $v \in [1, H]$). The seller then runs existing DSIC Bayesian mechanisms using D', while the players' values are drawn from D. In this sense, all our mechanisms are simple, but they are not given a true Bayesian instance as input.

The query algorithm $\mathcal{A}_V$ is defined in Algorithm 1. Here $D \in \Delta(\mathbb{R})$ is the distribution to be queried. The algorithm takes two parameters, the value bound H and the precision factor $\delta > 0$, makes $O(\log_{1+\delta} H)$ value queries to the oracle, and then returns a discrete distribution D'. It is easy to verify that D' is stochastically dominated by D.[5]

Denoting by $\mathcal{M}_{MRS}$ Myerson's mechanism for single-item auctions, Mechanism 2 defines our *efficient value Myerson* mechanism $\mathcal{M}_{EVM}$.

The query complexity of $\mathcal{M}_{EVM}$ is $O(n \log_{1+\epsilon} H)$, since each distribution D_i needs $O(\log_{1+\epsilon} H)$ value queries in $\mathcal{A}_V$. When ϵ is sufficiently small, $O(n \log_{1+\epsilon} H) \approx O(n\epsilon^{-1} \log H)$. Also, $\mathcal{M}_{EVM}$ is DSIC since $\mathcal{M}_{MRS}$ is so.

In this section and throughout the paper, we often analyze "mismatching" cases where a Bayesian mechanism $\mathcal{M}$ uses distribution D' while the actual Bayesian instance is $\mathcal{I} = (N, M, D)$ (i.e., the players' true values are drawn from D). We use $\mathrm{Rev}(\mathcal{M}(\mathcal{I}; D'))$ to denote the expected revenue in this case. By construction, $\mathrm{Rev}(\mathcal{M}_{EVM}(\mathcal{I})) = \mathrm{Rev}(\mathcal{M}_{MRS}(\mathcal{I}; D'))$.

Because the distribution D' constructed in $\mathcal{M}_{EVM}$ is stochastically dominated by D, letting $\mathcal{I}' = (N, M, D')$ be the Bayesian instance under D', by revenue monotonicity [22] we have $\mathrm{Rev}(\mathcal{M}_{MRS}(\mathcal{I}; D')) \geq \mathrm{Rev}(\mathcal{M}_{MRS}(\mathcal{I}'))$. By Lemma 5 of [22], $\mathrm{Rev}(\mathcal{M}_{MRS}(\mathcal{I}')) \geq \frac{\mathrm{OPT}(\mathcal{I})}{1+\epsilon}$. Thus we have the following simple result.

[5] Moreover, it is worth mentioning that the mean of D' approximates that of D. Indeed, $\mathrm{mean}(D) = \int_1^H v dF(v) \leq \sum_{l=0}^{k-1} v_{l+1} Pr[v_l \leq v < v_{l+1}] + v_k D(v_k) \leq (1+\delta) \sum_{l=0}^{k} v_l D'(v_l) = (1+\delta)\mathrm{mean}(D')$. Therefore, by directly applying the parametric mechanism in [2] with parameter $\mathrm{mean}(D')$ (for single-parameter auctions where the distributions are regular or MHR (monotone hazard rate)), we will get at least a $(1+\delta)$ fraction of their revenue.

Algorithm 3 The quantile-query algorithm $\mathcal{A}_Q$.

Input: the tail length ϵ_1 and the precision factor δ.
1: Let $k = \lceil \log_{1+\delta} \frac{1}{\epsilon_1} \rceil$ and define the *quantile vector* as $q = (q_0, q_1, \ldots, q_{k-1}, q_k) = (1, \epsilon_1(1+\delta)^{k-1}, \ldots, \epsilon_1(1+\delta), \epsilon_1)$.
2: Query the oracle for D with q, and receive a non-decreasing value vector $(v_l)_{l \in \{0,\ldots,k\}}$.
3: Construct a distribution D' as follows: $D'(v_l) = q_l - q_{l+1}$ for each $l \in \{0, \ldots, k\}$, where $q_{k+1} \triangleq 0$.
Output: Distribution D'.

Theorem 1. $\forall \epsilon > 0$, *for any single-item instance* $\mathcal{I} = (N, M, D)$ *with values in* $[1, H]$, *mechanism* $\mathcal{M}_{EVM}$ *is DSIC, has query complexity* $O(n \log_{1+\epsilon} H)$, *and* $\text{Rev}(\mathcal{M}_{EVM}(\mathcal{I})) \geq \frac{\text{OPT}(\mathcal{I})}{1+\epsilon}$.

3.2. Unbounded distributions and quantile-query mechanisms

Next, we construct efficient query mechanisms for arbitrary distributions whose supports can be unbounded. For a mechanism to approximate the optimal Bayesian revenue using *finite* non-adaptive queries to such distributions, it is intuitive that some kind of *small-tail assumption* for the distributions is needed. Indeed, given any mechanism with query complexity C, there always exists a distribution that has a sufficiently small probability mass around a sufficiently large value, such that the mechanism cannot find it using C queries. If this probability mass is where all the revenue comes (e.g., all the remaining probability mass is around value 0), then the query mechanism cannot be a good approximation to the optimal revenue. Following the literature [42,22], the small-tail assumptions are such that the expected revenue generated from the "tail" of the distributions is negligible compared to the optimal revenue; see Section 3.2.1. Distributions with bounded supports automatically satisfy these assumptions, so are regular distributions in single-item auctions.

Even with small-tail assumptions, it is hard to generate good revenue from unbounded distributions with finite *value* queries.[6] Instead, we show it is sufficient to use only *quantile* queries. As before, the seller uses our quantile-query algorithm $\mathcal{A}_Q$ (defined in Section 3.2.2) to learn a distribution D' that approximates D, and then reduces to simple mechanisms under D'. However, even for single-item auctions, it is not so simple to show why the combination of these two parts works. Indeed, under value queries it is easy to "under-price" the item so that the probability of sale is the same as in the optimal mechanism for D. Under quantile queries, under-pricing may lose a large amount of revenue because, for given quantiles, there is no guarantee on where the corresponding values are. Instead, the main idea in using quantile queries is to *"over-price"* the item. This is risky in many auction design scenarios, because it may significantly reduce the probability of sale, and thus lose a lot of revenue. We prove a key technical lemma in Lemma 2 for single-item auctions, where we show that by discretizing the quantile space properly, we can over-price the item while almost preserving the probability of sale as in the optimal mechanism under D. This guarantees that the revenue loss for using quantile queries is small.

3.2.1. Small-tail assumptions

A Bayesian auction instance $\mathcal{I}$ satisfies the *Small-Tail Assumption* if there exists a function[7] $h : (0, 1) \to (0, 1)$ such that, for any constant $\delta_1 \in (0, 1)$ and any BIC mechanism $\mathcal{M}$, letting $\epsilon_1 = h(\delta_1)$, we have[8]

$$\mathop{\mathbb{E}}_{v \sim D} \mathbf{I}_{\exists i,j, q_{ij}(v_{ij}) \leq \epsilon_1} \text{Rev}(\mathcal{M}(v; \mathcal{I})) \leq \delta_1 \text{OPT}(\mathcal{I}). \tag{1}$$

Here $q_{ij}(v_{ij})$ is the quantile of v_{ij} under distribution D_{ij}, $\text{Rev}(\mathcal{M}(v; \mathcal{I}))$ is the revenue of $\mathcal{M}$ under the Bayesian instance $\mathcal{I}$ when the true valuation profile is v, and $\mathbf{I}$ is the indicator function. For discrete distributions, Equation (1) is imposed on the ϵ_1 probability mass over the highest values. Intuitively, the small tail assumption assumes that the revenue contribution of any mechanism from high values, i.e., values with quantile smaller than ϵ, is sufficiently small. Similar assumptions are widely adopted in sampling mechanisms to deal with distributions with potentially unbounded supports [c.f. 22]. Examples of distributions satisfying small tail assumption include exponential distributions and all bounded distributions. See Section 6.3 for a detailed discussion.

3.2.2. The quantile-query algorithm

We define our quantile-query algorithm $\mathcal{A}_Q$ in Algorithm 3. As before, $D \in \Delta(\mathbb{R})$ is the distribution to be queried. The algorithm takes two parameters, the tail length ϵ_1 and the precision factor δ, makes $O(\log_{1+\delta} \frac{1}{\epsilon_1})$ quantile queries to the oracle, and then returns a discrete distribution D'.

3.2.3. Efficient quantile Myerson mechanism

Mechanism 4 defines our *efficient quantile Myerson* mechanism $\mathcal{M}_{EQM}$.

[6] Given any finite number of value queries, no information about the distribution is revealed if the distribution's support is above the maximum value query. Hence the query mechanism cannot distinguish among those distributions and constant approximations are not possible in this case.

[7] If computation complexity is a concern, then one can further require that the function is efficiently computable.

[8] In fact in our results for the single item auction, we only need a weaker condition where the inequality only need to hold for the Bayesian optimal mechanism OPT.

Mechanism 4 Efficient quantile Myerson mechanism $\mathcal{M}_{EQM}$.

1: Given $\epsilon > 0$, run algorithm $\mathcal{A}_Q$ with $\delta = \frac{\epsilon}{3}$ and $\epsilon_1 = h(\frac{2\epsilon}{3(1+\epsilon)})$ (i.e., $\delta_1 = \frac{2\epsilon}{3(1+\epsilon)}$ for Small Tail Assumption 2), for each player i's distribution D_i. Denote by D'_i the returned distribution. Let $D' = \times_{i\in N} D'_i$.
2: Run Myerson's mechanism $\mathcal{M}_{MRS}$ with D' and the players' reported values, $b = (b_i)_{i\in N}$, to get allocation $x = (x_i)_{i\in N}$ and price profile $p = (p_i)_{i\in N}$ as the outcome.

Theorem 2. *$\forall\epsilon > 0$, any single-item instance $\mathcal{I} = (N, M, D)$ satisfying Small-Tail Assumption, $\mathcal{M}_{EQM}$ is DSIC, has query complexity $O(-n \log_{1+\frac{\epsilon}{3}} h(\frac{2\epsilon}{3(1+\epsilon)}))$, and $\mathrm{Rev}(\mathcal{M}_{EQM}(\mathcal{I})) \geq \frac{\mathrm{OPT}(\mathcal{I})}{1+\epsilon}$.*

Before proving Theorem 2, we first claim the following key lemma via an imaginary Bayesian mechanism that "over-prices". Recall $\mathcal{I}' = (N, M, D')$ is the instance under D'.

Lemma 2. $\mathrm{Rev}(\mathcal{M}_{MRS}(\mathcal{I}')) \geq \frac{1}{1+\epsilon}\mathrm{OPT}(\mathcal{I})$.

Proof. For each player i, denote the support of D'_i by $V'_i = (v'_{i;l})_{l\in\{0,\dots,k\}}$. We first define a way to couple the values $v'_i \sim D'_i$ with the values $v_i \sim D_i$.

The randomized round-down scheme For any value $v_i \geq v'_{i;0}$, let v_i^- be v_i *rounded down* to the support of D'_i, such that v_i^- is distributed according to D'_i whenever v_i is distributed according to D_i. Recall that under value queries, v_i^- is simply the largest value in V'_i that is less than or equal to v_i, no matter whether D_i is continuous or discrete. Under quantile queries, when D_i is continuous, the same deterministic round-down scheme still works. However, the situation is more subtle when D_i is discrete, and we need a randomized round-down scheme to ensure the relationship between v_i and v_i^-. More precisely, by the definition of quantile queries, V'_i is a subset of D_i's support. If v_i is not in V'_i, then it is still deterministically rounded down as before. If v_i is in V'_i, say $v_i = v'_{i;l}$, then by the definition of quantile queries and the construction of D'_i, we have $\Pr_{x\sim D_i}[x \geq v_i] \geq q_l = \Pr_{x\sim D'_i}[x \geq v_i]$. In this case, v_i is rounded down to $v'_{i;l-1}$ (i.e., $v_i^- = v'_{i;l-1}$) with probability

$$\frac{\Pr_{x\sim D_i}[x \geq v_i] - \Pr_{x\sim D'_i}[x \geq v_i]}{D_i(v_i)},$$

and to $v'_{i;l}$ (i.e., $v_i^- = v'_{i;l}$) with probability

$$1 - \frac{\Pr_{x\sim D_i}[x \geq v_i] - \Pr_{x\sim D'_i}[x \geq v_i]}{D_i(v_i)}.$$

Following this scheme, it is not hard to verify that $\Pr_{v_i\sim D_i}[v_i^- \geq v'_{i;l}] = q_l$ for any $l \in \{0, \dots, k\}$, thus v_i^- is distributed according to D'_i, as desired.

No matter what v_i^- is, let v_i^+ be the smallest value in V'_i that is strictly larger than v_i^- (if no such value exists, then $v_i^+ = +\infty$). That is, $v_i^+ \geq v_i$ and v_i^+ is v_i "rounded up", which was not needed under value queries and is new for quantile queries.

The randomized resampling scheme For any value $v'_i \sim D'_i$, let v_i be *resampled* from D_i conditional on "v_i rounded down to v'_i", so that v_i is distributed according to D_i whenever v'_i is distributed according to D'_i. Again, under value queries, the resampling is simply conditional on $v_i \in [v'_{i;l}, v'_{i;l+1})$ when $v'_i = v'_{i;l}$, no matter whether D_i is continuous or discrete. Under quantile queries, this resampling scheme still works when D_i is continuous. When D_i is discrete, we need to "undo" the randomized round-down scheme defined above. More precisely, letting $v'_i = v'_{i;l}$, v_i is set to be $v'_{i;l+1}$ with probability

$$p_1 = \frac{\Pr_{x\sim D_i}[x \geq v'_{i;l+1}] - q_{l+1}}{D'_i(v'_{i;l})};$$

is resampled from D_i conditional on $v_i \in (v'_{i;l}, v'_{i;l+1})$ with probability

$$p_2 = \frac{\Pr_{x\sim D_i}[v'_{i;l} < x < v'_{i;l+1}]}{D'_i(v'_{i;l})};$$

and is set to be $v'_{i;l}$ with probability

$$p_3 = \frac{\Pr_{x\sim D_i}[x \leq v'_{i;l}] - \Pr_{x\sim D'_i}[x < v'_{i;l}]}{D'_i(v'_{i;l})} = \frac{D_i(v'_{i;l}) - \Pr_{x\sim D_i}[x \geq v'_{i;l}] + q_l}{D'_i(v'_{i;l})}.$$

Following this resampling scheme, it is not hard to verify that v_i is distributed according to D_i whenever v'_i is distributed according to D'_i.

Mechanism 5 A Bayesian mechanism $\mathcal{M}^*$ for instance $\mathcal{I}'$.

1: Each player i reports his value v'_i, and the mechanism discards the report that is not in V'_i.
2: For each player i, generate value v_i according to v'_i using our resampling scheme.
3: Run $\mathcal{M}_{MRS}$ with the value profile v and the prior distribution D, to get the price p_i and the allocation $x_i \in \{0, 1\}$ for each player i.
4: If $x_i = 1$ and $p_i \le v'_i$, sell the item to i and charge him p_i; otherwise, set $x_i = 0$ and $p_i = 0$.

Given the round-down and the resampling schemes above, we consider the Bayesian mechanism $\mathcal{M}^*$ defined in Mechanism 5 for $\mathcal{I}'$, and compare its revenue with that of $\mathcal{M}_{MRS}$. We first claim that $\mathcal{M}^*$ is a DSIC mechanism. Because $\mathcal{M}_{MRS}$ is DSIC, each x_i is monotone in v_i. Although v_i is a random variable given v'_i, it is easy to see that for any two different values $v'_i \in V'_i$ and $\hat{v}'_i \in V'_i$, the corresponding resampled values v_i and $\hat{v}_i$ are such that $v'_i < \hat{v}'_i$ implies $v_i \le \hat{v}_i$. Thus x_i is monotone in v'_i as well. Moreover, let θ_i be player i's threshold payment in $\mathcal{M}_{MRS}$ given v_{-i} and D. If $v'_i > \theta_i$ then $v_i > \theta_i$, thus player i gets the item at price $p_i = \theta_i$. If $v'_i < \theta_i$ then player i does not get the item and $p_i = 0$, no matter whether $v_i < \theta_i$ or not. Accordingly, θ_i is also player i's threshold payment in $\mathcal{M}^*$ under v_{-i} and D'. Since v_{-i} does not depend on v'_i, $\mathcal{M}^*$ is DSIC as desired.

To analyze the revenue of $\mathcal{M}^*$, note that by construction, when v'_i is distributed according to D'_i, the resampled v_i in $\mathcal{M}^*$ is distributed according to D_i. Moreover, each v'_i is distributed as if we first sample v_i from D_i and then setting $v'_i = v_i^-$.

Thus, mechanism $\mathcal{M}^*$ on instance $\mathcal{I}'$ essentially generates the same expected revenue as $\mathcal{M}_{MRS}$ on instance $\mathcal{I}$, except for the case when $v'_i < p_i \le v_i$ for the winner i. Fortunately, we are able to upper-bound the probability of this event and thus upper-bound the expected revenue loss. More precisely, for each player i, we write p_i as $p_i(v_{-i}; D)$ to emphasize that it is the threshold payment for i given v_{-i} and D, and does not depend on v_i or v'_i. We have

$$\begin{aligned}\mathrm{Rev}(\mathcal{M}^*(\mathcal{I}')) &= \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i\sim D_i} p_i(v_{-i}; D)\mathbf{I}_{v_i^- \ge p_i(v_{-i};D)} \\ &= \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} p_i(v_{-i}; D) \cdot \Pr_{v_i\sim D_i}[v_i^- \ge p_i(v_{-i}; D)]. \end{aligned} \tag{2}$$

Here the first equality holds because of the relationship between D' and D as established by our rounding and resampling schemes, and because each player i in $\mathcal{M}^*$ pays the same threshold price as in mechanism $\mathcal{M}_{MRS}$ whenever v'_i is at least the threshold, and pays 0 otherwise. By the construction of the distribution D', we have the following claim, which is proved in Appendix A.

Claim 1. $\Pr_{v_i\sim D_i}[v_i \ge p_i(v_{-i}; D)|q_i(v_i) > \epsilon_1] \le (1+\delta)\Pr_{v_i\sim D_i}[v_i^- \ge p_i(v_{-i}; D)]$.

Combining Equation (2), Claim 1 and Small Tail Assumption 2, we are able to lower-bound the revenue of $\mathcal{M}^*$ as follows, which is also proved in Appendix A.

Claim 2. $\mathrm{Rev}(\mathcal{M}^*(\mathcal{I}')) \ge \frac{1}{1+\epsilon}\mathrm{OPT}(\mathcal{I})$.

By the optimality of $\mathcal{M}_{MRS}$, $\mathrm{Rev}(\mathcal{M}_{MRS}(\mathcal{I}')) \ge \mathrm{Rev}(\mathcal{M}^*(\mathcal{I}'))$, and Lemma 2 holds. □

Proof of Theorem 2. First, mechanism $\mathcal{M}_{EQM}$ is DSIC because $\mathcal{M}_{MRS}$ is DSIC. Second, the query complexity of $\mathcal{M}_{EQM}$ is $O(-n\log_{1+\frac{\epsilon}{3}} h(\frac{2\epsilon}{3(1+\epsilon)}))$, because there are $k+1 = \lceil \log_{1+\frac{\epsilon}{3}} \frac{1}{h(\frac{2\epsilon}{3(1+\epsilon)})} \rceil + 1$ quantile queries for each player and there are n players in total. By definition,

$\mathrm{Rev}(\mathcal{M}_{EQM}(\mathcal{I})) = \mathrm{Rev}(\mathcal{M}_{MRS}(\mathcal{I}; D'))$. By construction, D' is stochastically dominated by D. Thus by revenue monotonicity $\mathrm{Rev}(\mathcal{M}_{MRS}(\mathcal{I}; D')) \ge \mathrm{Rev}(\mathcal{M}_{MRS}(\mathcal{I}'))$. Combining these two equations with Lemma 2, Theorem 2 holds. □

Mechanism $\mathcal{M}_{EQM}$ and Theorem 2 immediately extend to single-parameter downward-closed settings. Finally, when the distributions are regular, we are able to prove an even better query complexity and a matching lower-bound; see Section 5.

3.3. *Robustness to noisy queries*

When the value queries or the quantile queries exhibit η-noise, the optimal revenue cannot be obtained even with infinitely many queries. However, we can show that the performance of the query mechanisms degrades proportionally to the amount of noise η. Next we illustrate the multiplicative loss for value queries, and the bounds for quantile queries hold analogously.

When the seller constructs the empirical distribution D' in Algorithm 1 with $O(n\log_{1+\epsilon} H)$ value queries, in addition to rounding down the value, the seller can also choose to round down the quantile by a multiplicative factor of $1+\eta$ to guarantee that the constructed empirical distribution D' is stochastically dominated by the true distribution D. Moreover, there exists another distribution $\bar{D}$ stochastically dominates the true distribution D, which is obtained by rounding up both the valuation by $1+\epsilon$ and the quantile by $1+\eta$. Moreover, by Lemma 5 of [22] and Lemma 2, the revenue gap between D' and $\bar{D}$ is at most $(1+\epsilon)(1+\eta)^2$. Combining the results in Section 3.1, letting $\bar{\mathcal{I}} = (N, M, \bar{D})$ be the instance under $\bar{D}$, we have

$$\begin{aligned}&\text{Rev}(\mathcal{M}_{MRS}(\mathcal{I}; D')) \geq \text{Rev}(\mathcal{M}_{MRS}(\mathcal{I}'))\\&\geq \frac{1}{(1+\epsilon)(1+\eta)^2}\text{Rev}(\mathcal{M}_{MRS}(\bar{\mathcal{I}})) \geq \frac{1}{(1+\epsilon)(1+\eta)^2}\text{Rev}(\mathcal{M}_{MRS}(\mathcal{I})).\end{aligned}$$

This additional gap of $(1+\eta)^2$ is precisely the revenue loss the seller suffers from η-noisy queries.

Finally, if the queries are only approximately accurate with high probability, i.e., there exists $\delta, \eta > 0$ such that with probability at least $1-\delta$, all queries have η-noise, then we can easily show that with probability at least $1-\delta$, the query mechanisms are an $(1+\epsilon)(1+\eta)^2$-approximation to the optimal revenue.

4. Lower bounds

In this section, we prove lower bounds for the query complexity of Bayesian mechanisms when the queries have no noise, and we focus on DSIC mechanisms. As a building block for our general lower bound, we first have the following for *single-item single-player* auctions. Note that although the main focus of this paper is about non-adaptive noisy queries, here we prove a stronger result by showing that even with adaptive queries and precise feedback, our bounds for *single-item single-player* auctions are still tight. Formally, we have the following lemma.

Lemma 3. *For any constant $c > 1$, there exists a constant C such that, for any large enough H, any Bayesian mechanism $\mathcal{M}$ making less than $C \log_c H$ adaptive value and quantile queries to the oracle, there exists a single-player single-item Bayesian auction instance $\mathcal{I} = (N, M, D)$ where the values are bounded in $[1, H]$, such that $\text{Rev}(\mathcal{M}(\mathcal{I})) < \frac{\text{OPT}(\mathcal{I})}{c}$.*

Proof. Here we consider the equal revenue curve, i.e., $F(v) = 1 - \frac{1}{v}, v \in [1, H)$ and $F(H) = 1$. For this distribution, posting any price from $[1, H]$ has exactly the same revenue 1. For any constant H, let $k \triangleq \lfloor \frac{1}{4} \log_{(4c)^{4c+2}} H \rfloor$. We divide the value interval $[1, H]$ into $k+1$ sub-intervals as follows: from right to left, $u_k = H$, and $u_s = \frac{u_{s+1}}{(4c)^{4c+2}}$ for each $s \in \{k-1, \ldots, 0\}$. For each value interval (u_s, u_{s+1}), there exists a corresponding quantile interval $(q_s, q_{s+1}) \triangleq (\frac{1}{u_{s+1}}, \frac{1}{u_s})$.

For each value $s \in \{k-1, \ldots, 0\}$, considering the pair of intervals (u_s, u_{s+1}) and (q_s, q_{s+1}), we construct $\lceil 4c \rceil$ distributions for it. More precisely, the distribution D^s_z for each $z \in [\lceil 4c \rceil]$ is defined as follows. For $v_z < u_s$ or $u_{s+1} < v_z < H$, the density of v_z is $\frac{1}{v_z^2}$. The probability of $v_z = H$ is $\frac{1}{H}$. The probability of $(4c)^z u_s$ is $q_{s+1} - q_s$.

By construction, given parameter s, all $[\lceil 4c \rceil]$ distributions coincide outside the value and quantile range (u_s, u_{s+1}) and (q_s, q_{s+1}). Therefore, only a query inside the range can distinguish those distributions from each other. Moreover, for any two parameter $s, s' \in \{k-1, \ldots, 0\}$, the value and quantile queries will get the exactly same response for queries outside their own value and quantile intervals. Thus, for any adaptive value and quantile queries, a single query can distinguish at most one set of $\lceil 4c \rceil$ distributions. Letting $c' \triangleq 1 - \frac{1}{2c}$ and $C \triangleq \frac{1-c'}{8(4c+2)\log_c(4c)} = \frac{1}{16c(4c+2)\log_c(4c)}$, we have $C \log_c H < k(1-c')$. Accordingly, for any Bayesian mechanism $\mathcal{M}$ that makes less than $C \log_c H$ adaptive value and quantile queries, there exists a value s such that, with probability at least c', $\mathcal{M}$ cannot distinguish D^s_z's from each other.

We now analyze the optimal BIC revenue for those instances. For any $\mathcal{I}_z = (N, M, D_z)$, Myerson's mechanism is optimal: it sets a (randomized) threshold for the unique player, if the player bids at least the threshold then he gets the item and pays the threshold payment, otherwise the item is unsold. It is not hard to verify that $\text{OPT}(\mathcal{I}_z) = (4c)^z u_s q_{s+1}$ for each $\mathcal{I}_z$.

Next, we analyze the revenue of $\mathcal{M}$. Since $\mathcal{M}$ is DSIC, the allocation rule must be monotone in the player's bid, and he will pay the threshold payment set by $\mathcal{M}$, denoted by P. Here P may also be randomized. Note that for all instances, setting $P < 4cu_s$ is strictly worse than setting $P = 4cu_s$, and setting $P > (4c)^{\lceil 4c \rceil} u_s$ is strictly worse than setting $P = (4c)^{\lceil 4c \rceil} u_s < u_{s+1}$. Also, for any instance $\mathcal{I}_z$ and any $z' \in \{1, \ldots, \lceil 4c \rceil - 1\}$, setting $P \in ((4c)^{z'} u_s, (4c)^{z'+1} u_s)$ is strictly worse than setting $P = (4c)^{z'+1} u_s$. Thus, when mechanism $\mathcal{M}$ cannot distinguish the $\mathcal{I}_z$'s, it must use the same P for all $\mathcal{I}_z$'s, and the best it can do is to set $P = (4c)^z u_s$ with some probability ρ_z for each $z \in [\lceil 4c \rceil]$. Because $\sum_{z \in [\lceil 4c \rceil]} \rho_z = 1$, there exists z^* such that $\rho_{z^*} \leq \frac{1}{4c}$. Thus we have

$$\begin{aligned}\text{Rev}(\mathcal{M}(\mathcal{I}_{z^*})) &\leq \frac{1}{4c} \cdot (4c)^{z^*} \cdot u_s \cdot q_{s+1} + (1 - \frac{1}{4c})(4c)^{z^*-1} \cdot u_s \cdot q_{s+1}\\&< \frac{1}{2c} \cdot (4c)^{z^*} \cdot u_s \cdot q_{s+1} = \frac{1}{2c}\text{OPT}(\mathcal{I}_{z^*}),\end{aligned}$$

where the first inequality is because for any threshold other than $(4c)^{z^*} u_s$, the resulting expected revenue is no larger than that with the threshold being $(4c)^{z^*-1} u_s$. That is, when $\mathcal{M}$ cannot distinguish the $\mathcal{I}_z$'s, it cannot be a $2c$-approximation for $\mathcal{I}_{z^*}$.

As the revenue of $\mathcal{M}$ under $\mathcal{I}_{z^*}$ is at most $\text{OPT}(\mathcal{I}_{z^*})$ when it is able to distinguish $\mathcal{I}_{z^*}$ from all the other instances, we have

$$\text{Rev}(\mathcal{M}(\mathcal{I}_{z^*})) \leq (1 - \frac{1}{2c})\frac{1}{2c}\text{OPT}(\mathcal{I}_{z^*}) + \frac{1}{2c}\text{OPT}(\mathcal{I}_{z^*}) < \frac{1}{c}\text{OPT}(\mathcal{I}_{z^*}).$$

Thus $\mathcal{M}$ is not a c-approximation for $\mathcal{I}_{z^*}$, and Lemma 3 holds. □

We extend Lemma 3 to arbitrary multi-player multi-item Bayesian auctions with *succinct sub-additive* valuations, as follows. To make our exposition clearer, we first introduce some notations. A very broad class of Bayesian auctions, *(monotone) sub-additive* auctions, is such that each player i has a valuation function $v_i : 2^{[m]} \to \mathbb{R}$, which satisfies $v_i(S) + v_i(T) \geq v_i(S \cup T) \geq v_i(S) \geq 0$ for any subsets of items S and T. As such a valuation function in general needs 2^m values to describe, following the conventions in Bayesian auction design [43,16,13], we will consider *succinct sub-additive* auctions, where only the item-values, that is, the v_{ij}'s, are independently drawn from the underlying distribution $D = \times_{i\in[n], j\in[m]} D_{ij}$. Given $(v_{ij})_{j\in[m]}$, it is publicly known how to compute player i's value for any subset of items. That is, the valuation function v_i now takes a vector of item-values $(v_{ij})_{j\in[m]}$ and a subset $S \subseteq [m]$ as inputs, such that for any vector $(v_{ij})_{j\in[m]}$, the resulting function $v_i((v_{ij})_{j\in[m]}, \cdot)$ is sub-additive and $v_i((v_{ij})_{j\in[m]}, \{j\}) = v_{ij}$ for each item j. Note that such auctions include single-item, unit-demand and additive auctions as special cases.

Theorem 3. *For any constant $c > 1$, there exists a constant C such that, for any $n \geq 1, m \geq 1$, for any large enough H, any monotone sub-additive valuation function profile $v = (v_i)_{i\in[n]}$, and any Bayesian mechanism $\mathcal{M}$ making less than $Cnm \log_{m \ln H} H$ adaptive value and quantile queries to the oracle, there exists a multi-item Bayesian auction instance $\mathcal{I} = (N, M, D)$, where $|N| = n$, $|M| = m$ and the values are bounded in $[1, H]$, such that $\text{Rev}(\mathcal{M}(\mathcal{I})) < \frac{\text{OPT}(\mathcal{I})}{c}$.*

Proof. Similar to the proof of Lemma 3, for any H, let $k \triangleq \lfloor \frac{1}{4} \log_{(4cx)^{4c+2}} H \rfloor$, where x is a parameter to be determined later. Let $C' \triangleq \frac{1}{24c(4c+2)\log_c(4cx)}$ and H be large enough so that $k \geq 1$. We divide the value interval $[1, H]$ into $k + 1$ sub-intervals as follows: from right to left, $u_k = H$, and $u_s = \frac{u_{s+1}}{(4cx)^{4c+2}}$ for each $s \in \{k - 1, \ldots, 0\}$. For each value interval (u_s, u_{s+1}), there exists a corresponding quantile interval $(q_s, q_{s+1}) \triangleq (\frac{1}{u_{s+1}}, \frac{1}{u_s})$.

It is easy to see that $C'nm \log_c H < \frac{nmk}{3c}$. Thus, for any Bayesian mechanism $\mathcal{M}$ making less than $C'nm \log_c H$ adaptive value and quantile queries, with probability at least $1 - \frac{1}{3c}$, there exists a player-item pair (i^*, j^*), a value interval (u_s, u_{s+1}) with corresponding quantile interval (q_s, q_{s+1}) such that $\mathcal{M}$ does not query these two intervals for $D_{i^*j^*}$ and does not distinguishes $D^z_{i^*j^*;s}$'s from each other.

Therefore, for each $i \in [n], j \in [m], s \in [k]$, we construct $\lceil 4c \rceil$ Bayesian instances $\{\mathcal{I}^z = (N, M, D^z)\}_{z\in[\lceil 4c \rceil]}$, where each D^z_{ij} is equal revenue distribution if $i \neq i^*$ or $j \neq j^*$. For any $z \in [\lceil 4c \rceil]$, we construct the distribution $D^z_{ij;s}$ such that for $v_z < u_s$ or $u_{s+1} < v_z < H$, the density of v_z is $\frac{1}{v_z^2}$. The probability of $v_z = H$ is $\frac{1}{H}$. The probability of $(4cx)^z u_s$ is $q_{s+1} - q_s$. All distributions other than D_{ij} are equal revenue distribution. For any queries outside the interval (u_s, u_{s+1}) and (q_s, q_{s+1}), it cannot distinguish those instances. Thus, with less than $Cnm \log_c H$ adaptive value and quantile queries, with probability at least $1 - \frac{1}{3c}$, the mechanism cannot distinguish those instances from each other.

We now analyze the optimal BIC revenue for those instances. For any $\mathcal{I}_z$, Myerson's mechanism is optimal: it sets a (randomized) threshold for the unique player, if the player bids at least the threshold then he gets the item and pays the threshold payment, otherwise the item is unsold. Letting $\delta \triangleq \frac{1}{H}$, it is not hard to verify that $\text{OPT}(\mathcal{I}_z) = (4cx)^z u_s q_s$ for each $\mathcal{I}_z$.

Given any succinct sub-additive valuation function profile $v = (v_i)_{i\in[n]}$ where each v_i takes a vector of item-values $(v_{ij})_{j\in[m]}$ as part of its input, we would like to compare the optimal revenue for the sub-additive instances defined by the $\mathcal{I}^z$'s with the corresponding expected revenue of $\mathcal{M}$. By construction, the D^z's differ only at the $D^z_{i^*j^*}$'s, within the value interval (u_s, u_{s+1}) and the quantile interval (q_s, q_{s+1}). Accordingly, with probability at least $1 - \frac{1}{3c}$, mechanism $\mathcal{M}$ cannot distinguish the $\mathcal{I}^z$'s from each other. Eventually, we will analyze the revenue of $\mathcal{M}$ conditional on this event happening.

For now, to compare the optimal revenue and that of $\mathcal{M}$, let us first introduce some notations. For any item-value profile $\hat{v} = (\hat{v}_{ij})_{i\in[n], j\in[m]}$, when the players bid $\hat{v}$, we denote by $x_i(\hat{v})$ the (randomized) allocation of $\mathcal{M}$ to a player i. It is defined by the probabilities $\sigma_{iS}(\hat{v})$ for all the subsets $S \subseteq [m]$: each $\sigma_{iS}(\hat{v})$ is the probability that player i receives S under bid $\hat{v}$. Accordingly, the expected value of player i for allocation $x_i(\hat{v})$ is $v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})) = \sum_S v_i((\hat{v}_{ij})_{j\in[m]}, S) \cdot \sigma_{iS}(\hat{v})$. Moreover, for each item j, let $x_{ij}(\hat{v})$ be the probability that player i receives item j according to $x_i(\hat{v})$: that is, $x_{ij}(\hat{v}) = \sum_{S: j\in S} \sigma_{iS}(\hat{v})$.

We upper-bound the revenue of $\mathcal{M}$ in three steps. To begin with, we reduce the multi-player sub-additive instances to *single-player* sub-additive instances, and construct a DSIC Bayesian mechanism $\mathcal{M}^*$ that only sells the items to player i^*. Given any instance $\mathcal{I}^z$, mechanism $\mathcal{M}^*$ runs on the single-player sub-additive instance $\mathcal{I}^z_{i^*} = (\{i^*\}, M, D^z_{i^*})$. It first simulates the item values of players in $N \setminus \{i^*\}$, which are all 1's, and then runs $\mathcal{M}$. Mechanism $\mathcal{M}^*$ answers the oracle queries of $\mathcal{M}$ truthfully. The allocation and the payment for player i^* under $\mathcal{M}^*$ is the same as those under $\mathcal{M}$. For any player $i \neq i^*$, mechanism $\mathcal{M}^*$ assigns nothing to him and charges him 0, because i is an imaginary player to $\mathcal{M}^*$. It is easy to see that mechanism $\mathcal{M}^*$ is DSIC. Moreover,

$$\text{Rev}(\mathcal{M}^*(\mathcal{I}^z_{i^*})) \geq \text{Rev}(\mathcal{M}(\mathcal{I}^z)) - \mathbb{E}_{\hat{v}\sim D^z} \sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})), \tag{3}$$

because the revenue generated by $\mathcal{M}$ from players in $N \setminus \{i^*\}$ is at most their total value for the allocation.

Next, we reduce the single-player sub-additive instances to single-player *additive* instances, and construct a DSIC Bayesian mechanism $\mathcal{M}^+$ that runs on the single-player additive instances $\mathcal{I}^{+z}_{i^*} = (\{i^*\}, M, D^z_{i^*})$, with $z \in [\lceil 4c \rceil]$. Note that each $\mathcal{I}^{+z}_{i^*}$ has the same item-value distributions as $\mathcal{I}^z_{i^*}$, but player i^*'s value for any subset of items is additive.

For each single-player sub-additive instance defined by $\mathcal{I}^z_{i^*}$ and the valuation function profile v, by the taxation principle [28,40], mechanism $\mathcal{M}^*$ is equivalent to providing a menu of options, where each entry is a pair of bundle and its price, to player i^*, and then letting i^* choose an entry maximizing his expected utility according to his true valuation. Given any instance $\mathcal{I}^{+z}_{i^*}$, mechanism $\mathcal{M}^+$ provides the same menu as mechanism $\mathcal{M}^*$ under $\mathcal{I}^z_{i^*}$ and v, except that the payment in each entry is discounted by a multiplicative $1-\hat{\epsilon}$. Here $\hat{\epsilon}$ is a sufficiently small constant in $(0,1)$ to be determined later in the analysis. The truthfulness of $\mathcal{M}^+$ is immediate, because it lets i^* choose a menu entry maximizing his expected utility under his true additive values. Let

$$\bar{\delta} \triangleq \mathbb{E}_{\hat{v}_{i^*}\sim D^z_{i^*}} \max_{S\subseteq[m]} (\sum_{j\in S} \hat{v}_{i^*j} - v_{i^*}((\hat{v}_{i^*j})_{j\in[m]}, S)),$$

the expected maximum difference between the additive values and the succinct sub-additive values. Following Lemma 3.4 in [43], which compares the revenue in the sub-additive instance with that in the corresponding additive instance, we have

$$\text{Rev}(\mathcal{M}^+(\mathcal{I}^{+z}_{i^*})) \geq (1-\hat{\epsilon})(\text{Rev}(\mathcal{M}^*(\mathcal{I}^z_{i^*})) - \bar{\delta}/\hat{\epsilon}). \tag{4}$$

Finally, we reduce the single-player additive instances to single-player *single-item* instances, and consider a DSIC Bayesian mechanism $\mathcal{M}'$ that only sells item j^* to player i^*. Mechanism $\mathcal{M}'$ runs on the single-player single-item instances $\mathcal{I}^z_{i^*j^*} = (\{i^*\}, \{j^*\}, D^z_{i^*j^*})$, with $z \in [\lceil 4c \rceil]$. Given any $\mathcal{I}^z_{i^*j^*}$, it first lets player i^* report $\hat{v}_{i^*j^*}$. Then it simulates the $\hat{v}_{i^*j}$'s from $D^z_{i^*j}$ for $j \neq j^*$, which are all 1's, and runs $\mathcal{M}^+$ on the augmented additive instance $\mathcal{I}^{+z}_{i^*}$ to obtain allocation $x^+_{i^*}(\hat{v}_{i^*})$ and payment $p^+_{i^*}(\hat{v}_{i^*})$. For each item j, let $x^+_{i^*j}(\hat{v}_{i^*})$ be the probability that player i^* receives item j in the allocation. Mechanism $\mathcal{M}'$ sets its outcome to be the following:

- $x'_{i^*j^*}(\hat{v}_{i^*j^*}) = x^+_{i^*j^*}(\hat{v}_{i^*})$; and
- $p'_{i^*}(\hat{v}_{i^*j^*}) = p^+_{i^*}(\hat{v}_{i^*}) - \sum_{j\in[m]\setminus\{j^*\}} \hat{v}_{i^*j} x^+_{i^*j}(\hat{v}_{i^*})$.

Note that $p'_{i^*}(\hat{v}_{i^*j^*})$ may be negative. Moreover, mechanism $\mathcal{M}'$ is also considered in Lemma 8 of [32], which proved that $\mathcal{M}'$ is DSIC and

$$\text{Rev}(\mathcal{M}'(\mathcal{I}^z_{i^*j^*})) \geq \text{Rev}(\mathcal{M}^+(\mathcal{I}^{+z}_{i^*})) - \sum_{j\neq j^*} \mathbb{E}_{\hat{v}_{i^*j}\sim D^z_{i^*j}} \hat{v}_{i^*j}. \tag{5}$$

Now we combine the above three reduction steps together and consider the event when mechanism $\mathcal{M}$ cannot distinguish the $\mathcal{I}^z$'s from each other. When this happens, mechanism $\mathcal{M}$ produces the same outcome for all the instances. Accordingly, although mechanism $\mathcal{M}^*$ is given the distributions $D^z_{i^*}$, by simulating $\mathcal{M}$, it still produces the same outcome for all the $\mathcal{I}^z_{i^*}$'s, thus the same menu for all of them. So mechanism $\mathcal{M}^+$ also produces the same menu for all the $\mathcal{I}^{+z}_{i^*}$'s: that is, the menu produced by $\mathcal{M}^*$ with the payments discounted by $1-\hat{\epsilon}$. As a result, although mechanism $\mathcal{M}'$ is given the $D^z_{i^*j^*}$'s, it still cannot "distinguish" the $\mathcal{I}^z_{i^*j^*}$'s from each other and produces the same outcome for all of them. Using the similar argument in the proof of Lemma 3, in this case there exists $z^* \in [\lceil 4c \rceil]$ such that

$$\text{Rev}(\mathcal{M}'(\mathcal{I}^{z^*}_{i^*j^*})) < \frac{1}{2c}\text{OPT}(\mathcal{I}^{z^*}_{i^*j^*}).$$

Combining this inequality with Equations (3), (4) and (5), we have

$$\begin{aligned}
&\text{Rev}(\mathcal{M}(\mathcal{I}^{z^*})) \leq \text{Rev}(\mathcal{M}^*(\mathcal{I}^{z^*}_{i^*})) + \mathbb{E}_{\hat{v}\sim D^{z^*}} \sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})) \\
&\leq \frac{\text{Rev}(\mathcal{M}^+(\mathcal{I}^{+z^*}_{i^*}))}{1-\hat{\epsilon}} + \bar{\delta}/\hat{\epsilon} + \mathbb{E}_{\hat{v}\sim D^{z^*}} \sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v})) \\
&\leq \frac{1}{1-\hat{\epsilon}} \left(\text{Rev}(\mathcal{M}'(\mathcal{I}^{z^*}_{i^*j^*})) + \sum_{j\neq j^*} \mathbb{E}_{\hat{v}_{i^*j}\sim D^{z^*}_{i^*j}} \hat{v}_{i^*j} \right) + \bar{\delta}/\hat{\epsilon} + \mathbb{E}_{\hat{v}\sim D^{z^*}} \sum_{i\neq i^*} v_i((\hat{v}_{ij})_{j\in[m]}, x_i(\hat{v}))
\end{aligned}$$

$$< \frac{1}{1-\hat{\epsilon}} \left(\frac{1}{2c} \mathrm{OPT}(\mathcal{I}^{z^*}_{i^* j^*}) + \sum_{j \neq j^*} \mathbb{E}_{\hat{v}_{i^* j} \sim D^{z^*}_{i^* j}} \hat{v}_{i^* j} \right) + \bar{\delta}/\hat{\epsilon} + \mathbb{E}_{\hat{v} \sim D^{z^*}} \sum_{i \neq i^*} v_i((\hat{v}_{ij})_{j \in [m]}, x_i(\hat{v})). \tag{6}$$

Note that $\mathrm{OPT}(\mathcal{I}^{z^*}_{i^* j^*}) \leq \mathrm{OPT}(\mathcal{I}^{z^*})$, since selling a single item to a single player is a feasible outcome. Moreover, since $D^{z^*}_{ij}$ is constantly 1 when $i \neq i^*$ or $j \neq j^*$, and since the valuation function profile v is succinct sub-additive, we have

$$\sum_{j \neq j^*} \mathbb{E}_{\hat{v}_{i^* j} \sim D^{z^*}_{i^* j}} \hat{v}_{i^* j} = (m-1) \ln H,$$

$$\bar{\delta}/\hat{\epsilon} = \frac{1}{\hat{\epsilon}} \mathbb{E}_{\hat{v}_{i^*} \sim D^{z^*}_{i^*}} \max_{S \subseteq [m]} (\sum_{j \in S} \hat{v}_{i^* j} - v_{i^*}((\hat{v}_{i^* j})_{j \in [m]}, S)) \leq \frac{m \ln H}{\hat{\epsilon}},$$

$$\mathbb{E}_{\hat{v} \sim D^{z^*}} \sum_{i \neq i^*} v_i((\hat{v}_{ij})_{j \in [m]}, x_i(\hat{v})) \leq m \ln H.$$

Here the second equation is because $\sum_{j \in S} \hat{v}_{i^* j} - v_{i^*}((\hat{v}_{i^* j})_{j \in [m]}, S) \leq \sum_{j \in [m]} \hat{v}_{i^* j}$ for any $\hat{v}_{i^*}$ and S. The third equation is because $\sum_{i \neq i^*} v_i((\hat{v}_{ij})_{j \in [m]}, x_i(\hat{v})) \leq \sum_j \sum_{i \neq i^*} x_{ij}(\hat{v}) \hat{v}_{ij}$ for any $\hat{v}$: indeed, each item can be sold to at most one player, generating expected value $\ln H$.

Combining the equations above with Equation (6), we have

$$\mathrm{Rev}(\mathcal{M}(\mathcal{I}^{z^*})) < \frac{1}{1-\hat{\epsilon}} (\frac{1}{2c} \mathrm{OPT}(\mathcal{I}^{z^*}) + (m-1) \ln H) + \frac{m \ln H}{\hat{\epsilon}} + m \ln H.$$

Setting $\hat{\epsilon} = \frac{1}{4}$, we have

$$\mathrm{Rev}(\mathcal{M}(\mathcal{I}^{z^*})) < \frac{2}{3c} \mathrm{OPT}(\mathcal{I}^{z^*}) + \frac{19 m \ln H}{3}. \tag{7}$$

Now we combine Equation (7) with the probability that $\mathcal{M}$ cannot distinguish the $\mathcal{I}^z$'s. As $\mathrm{OPT}(\mathcal{I}^{z^*}) \geq 4cx$, when $x > \frac{57 m \cdot c \cdot \ln H}{8}$, we have $\mathrm{OPT}(\mathcal{I}^{z^*}) > \frac{57}{2} m \cdot c^2 \cdot \ln H$ and

$$\mathrm{Rev}(\mathcal{M}(\mathcal{I}^{z^*})) \leq (1 - \frac{1}{3c})(\frac{2}{3c} \mathrm{OPT}(\mathcal{I}^{z^*}) + \frac{19 m \ln H}{3}) + \frac{1}{3c} \mathrm{OPT}(\mathcal{I}^{z^*}) < \frac{1}{c} \mathrm{OPT}(\mathcal{I}^{z^*}).$$

Letting $C = \frac{1}{24c(4c+2)}$, which is a constant with respect to c, we have $Cnm \log_{m \ln H} H \leq C' nm \log_c H$. Thus finishes the proof of Theorem 3. □

When there are multiple items to sell, for non-adaptive queries, we can make some changes to the constructed distributions to improve the query complexity bound. Formally, consider the $\lceil 4c \rceil$ Bayesian instances $\{\mathcal{I}^z = (N, M, D^z)\}_{z \in [\lceil 4c \rceil]}$ that are indistinguishable using only $Cnm \log_c H$ non-adaptive queries. Here for $i \neq i^*$ or $j \neq j^*$, D^z_{ij} is the distribution that is constantly 1. For distribution $D^z_{i^* j^*}$, we construct them by shifting them in the interval with no value or quantile queries, i.e. (u_s, u_{s+1}) and (q_s, q_{s+1}), based on the equal revenue distribution with cumulative probability function $\max\{0, 1 - \frac{\sqrt{H}}{v}\}$. Note that the optimal revenue for those distributions is at least $\sqrt{H}$. Applying the same analysis as in Theorem 3, we have the following theorem, with proof omitted.

Theorem 4. *For any constant $c > 1$, there exists a constant C such that, for any $n \geq 1, m \geq 1$, any large enough H, any succinct sub-additive valuation function profile $v = (v_i)_{i \in [n]}$, and any DSIC Bayesian mechanism $\mathcal{M}$ making less than $Cnm \log_c H$ non-adaptive value and quantile queries to the oracle, there exists a multi-item Bayesian auction instance $\mathcal{I} = (N, M, D)$ with valuation profile v, where $|N| = n$, $|M| = m$ and the item values are bounded in $[1, H]$, such that* $\mathrm{Rev}(\mathcal{M}(\mathcal{I})) < \frac{\mathrm{OPT}(\mathcal{I})}{c}$.

Succinct sub-additive valuations is a very broad class and contains single-item, unit-demand, and additive auctions as special cases. Thus Theorem 3 and 4 automatically apply to those cases. We also note that it is shown in [45] that the optimal BIC revenue exceeds the optimal DSIC revenue by a constant factor even for two i.i.d. additive players and two identical items. So even with infinite samples, there exist constants $c > 1$ such that no c-approximation to OPT is possible. However, Theorem 4 is stronger: for *every* constant $c > 1$, one needs at least the given number of queries to get a c-approximation.

5. Extension I: single-item auctions with regular distributions

In this section, we show that when we only consider regular distributions for single-item auctions, the query complexity can be much lower. In fact, we no longer need the small-tail assumptions explicitly even when the supports are unbounded.

Mechanism 6 Efficient quantile Myerson mechanism for regular distributions, $\mathcal{M}_{EMR}$.

1: Given $\epsilon > 0$, run algorithm $\mathcal{A}_Q$ with $\delta = \frac{\epsilon}{4}$ and $\epsilon_1 = \frac{\epsilon^2}{256n}$ for each player i's distribution D_i, with the returned distribution denoted by D'_i. Let $D' = \times_{i\in N} D'_i$.
2: Run $\mathcal{M}_{MRS}$ with D' and the players' reported values, $b = (b_i)_{i\in N}$, to get allocation $x = (x_i)_{i\in N}$ and price profile $p = (p_i)_{i\in N}$ as the outcome.

More precisely, for the upper-bound, we show that regular distributions satisfy the small-tail property with a properly defined tail function. Thus our techniques for distributions with small-tails directly apply here.

For the lower-bound, recall that in Section 4 we allow the distributions to be irregular. To construct the desired distributions, we can first find the un-queried quantile interval and then move the probability mass from its end points to internal points. Because the distributions can be irregular, we have complete control on where to put the probability mass. However, if the distributions have to be regular then this cannot be done. Instead, we start from two different single-peaked revenue curves and construct regular distributions from them. We still want to move probability mass from the end points of the un-queried quantile interval to internal points, but such moves must be continuous in order to preserve regularity. Finally, our lower- and upper-bounds are *tight* upto a logarithmic factor.

5.1. Upper bound

Our mechanism $\mathcal{M}_{EMR}$ (i.e., "Efficient quantile Myerson mechanism for Regular distributions") first constructs the distribution D' that approximates D using the quantile-query algorithm $\mathcal{A}_Q$ with parameters $\delta = \frac{\epsilon}{4}$ and $\epsilon_1 = \frac{\epsilon^2}{256n}$; and then runs Myerson's mechanism $\mathcal{M}_{MRS}$ on D'. Formally, we have the following theorem.

Theorem 5. $\forall \epsilon \in (0,1)$*, and for any single-item instance* $\mathcal{I} = (N, M, D)$ *where* D *is regular, mechanism* $\mathcal{M}_{EMR}$ *is DSIC, has query complexity* $O(n \log_{1+\epsilon} \frac{n}{\epsilon})$*, and* $\text{Rev}(\mathcal{M}_{EMR}(\mathcal{I})) \geq \frac{\text{OPT}(\mathcal{I})}{1+\epsilon}$.

Proof. Consider the quantile value $q^* = \frac{\epsilon^2}{256n}$ and $v_i^* = F_i^{-1}(1-q^*)$. Let $\hat{v}_i = \max\{v_i^*, \frac{16\text{OPT}(\mathcal{I})}{\epsilon}\}$, and $\bar{D}_1, \ldots, \bar{D}_n$ be imaginary distributions obtained by truncating $D_1, \ldots, D_n$ at $\hat{v}_i$ (i.e., a sample $\bar{v}_i$ from $\bar{D}_i$ is obtained by first sampling v_i from D_i and then rounding down to $\bar{v}_i = \min\{v_i, \hat{v}_i\}$). Finally, denote by $\bar{\mathcal{I}} = (N, M, \bar{D})$ the imaginary Bayesian instance where players' values are drawn from $\bar{D}$.

Note that D' is also a discretization distribution for $\bar{D}$, following the proof and notations of Theorem 2, letting v_i^- be the value first sampled from $\bar{D}_i$ then rounding down to the support of D', we have $\mathcal{M}_{EMR}$ is truthful and using the technique of Mechanism 5, we have

$$\begin{aligned}
&\text{Rev}(\mathcal{M}_{EMR}(\mathcal{I})) = \text{Rev}(\mathcal{M}_{MRS}(v, D')) \geq \text{Rev}(\mathcal{M}_{MRS}(v', D')) \\
&\geq \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i}\sim\bar{D}_{-i}} p_i(\bar{v}_{-i}; \bar{D}) \cdot \Pr_{\bar{v}_i\sim\bar{D}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{D})] \\
&= \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i}\sim\bar{D}_{-i}} p_i(\bar{v}_{-i}; \bar{D}) \cdot \Pr_{\bar{v}_i\sim\bar{D}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{D})] \cdot (\mathbf{I}_{v_i^* \leq \frac{16\text{OPT}(\mathcal{I})}{\epsilon}} + \mathbf{I}_{v_i^* > \frac{16\text{OPT}(\mathcal{I})}{\epsilon}}).
\end{aligned} \tag{8}$$

We bound the indicators separately.

$$\begin{aligned}
&\sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i}\sim\bar{D}_{-i}} p_i(\bar{v}_{-i}; \bar{D}) \cdot \Pr_{\bar{v}_i\sim\bar{D}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{D})] \cdot \mathbf{I}_{v_i^* \leq \frac{16\text{OPT}(\mathcal{I})}{\epsilon}} \\
&= \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i}\sim\bar{D}_{-i}} p_i(\bar{v}_{-i}; \bar{D}) \cdot \Pr_{\bar{v}_i\sim\bar{D}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{D})] \cdot \mathbf{I}_{v_i^* \leq \frac{16\text{OPT}(\mathcal{I})}{\epsilon}} \cdot (\mathbf{I}_{p_i(\bar{v}_{-i};\bar{D}) < v_i^*} + \mathbf{I}_{p_i(\bar{v}_{-i};\bar{D}) \geq v_i^*}) \\
&\geq \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i}\sim\bar{D}_{-i}} [p_i(\bar{v}_{-i}; \bar{D}) \cdot \frac{1}{1+\frac{\epsilon}{4}} \cdot \Pr_{\bar{v}_i\sim\bar{D}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{D})] \cdot \mathbf{I}_{v_i^* \leq \frac{16\text{OPT}(\mathcal{I})}{\epsilon}} \cdot \mathbf{I}_{p_i(\bar{v}_{-i};\bar{D}) < v_i^*} \\
&\quad + (p_i(\bar{v}_{-i}; \bar{D}) \cdot \Pr_{\bar{v}_i\sim\bar{D}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{D})] - \frac{16\text{OPT}(\mathcal{I})}{\epsilon} \cdot \frac{\epsilon^2}{256n}) \cdot \mathbf{I}_{v_i^* \leq \frac{16\text{OPT}(\mathcal{I})}{\epsilon}} \cdot \mathbf{I}_{p_i(\bar{v}_{-i};\bar{D}) \geq v_i^*}] \\
&\geq \frac{1}{1+\frac{\epsilon}{4}} \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i}\sim\bar{D}_{-i}} p_i(\bar{v}_{-i}; \bar{D}) \cdot \Pr_{\bar{v}_i\sim\bar{D}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{D})] \cdot \mathbf{I}_{v_i^* \leq \frac{\text{OPT}(\mathcal{I})}{16\epsilon}} - \frac{\epsilon}{16} \cdot \text{OPT}(\mathcal{I}).
\end{aligned} \tag{9}$$

The first inequality here holds because for price $p_i(\bar{v}_{-i}; \bar{D}) < v_i^*$, we have

$$\Pr_{\bar{v}_i\sim\bar{D}_i}[v_i^- \geq p_i(\bar{v}_{-i}; \bar{D})] \geq \frac{1}{1+\frac{\epsilon}{4}} \Pr_{\bar{v}_i\sim\bar{D}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i}; \bar{D})]$$

due to the structure of the quantile queries for D'. For price $p_i(\bar{v}_{-i};\bar{D}) \geq v_i^*$, when $v_i^* \leq \frac{16\mathrm{OPT}(\mathcal{I})}{\epsilon}$, by the regularity of D_i, the optimal reserve corresponds to the quantile interval $(\frac{\epsilon^2}{256n}, 1]$. Thus we have

$$
\begin{aligned}
& p_i(\bar{v}_{-i};\bar{D}) \cdot \Pr_{\bar{v}_i \sim \bar{D}_i}[v_i^- \geq p_i(\bar{v}_{-i};\bar{D})] \geq 0 \\
\geq\ & p_i(\bar{v}_{-i};\bar{D}) \cdot \Pr_{\bar{v}_i \sim \bar{D}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i};\bar{D})] - v_i^* \cdot \Pr_{\bar{v}_i \sim \bar{D}_i}[\bar{v}_i \geq v_i^*] \\
\geq\ & p_i(\bar{v}_{-i};\bar{D}) \cdot \Pr_{\bar{v}_i \sim \bar{D}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i};\bar{D})] - \frac{16\mathrm{OPT}(\mathcal{I})}{\epsilon} \cdot \frac{\epsilon^2}{256n}
\end{aligned}
$$

since the expected revenue is non-decreasing for quantile range $[0, \frac{\epsilon^2}{256n}]$. Thus Equation (9) holds. Then for the second indicator for Equation (8), we have

$$
\begin{aligned}
& \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{D}_{-i}} p_i(\bar{v}_{-i};\bar{D}) \cdot \Pr_{\bar{v}_i \sim \bar{D}_i}[v_i^- \geq p_i(\bar{v}_{-i};\bar{D})] \cdot \mathbf{I}_{v_i^* > \frac{16\mathrm{OPT}(\mathcal{I})}{\epsilon}} \\
\geq\ & \frac{1}{1+\frac{\epsilon}{4}} \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{D}_{-i}} p_i(\bar{v}_{-i};\bar{D}) \cdot \Pr_{\bar{v}_i \sim \bar{D}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i};\bar{D})] \cdot \mathbf{I}_{v_i^* > \frac{16\mathrm{OPT}(\mathcal{I})}{\epsilon}}
\end{aligned} \tag{10}
$$

also by the construction of the quantile queries for D'. Combining Equation (8), (9) and (10), we have

$$
\begin{aligned}
& \mathrm{Rev}(\mathcal{M}_{EMR}(\mathcal{I})) \\
\geq\ & \frac{1}{1+\frac{\epsilon}{4}} \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{D}_{-i}} p_i(\bar{v}_{-i};\bar{D}) \cdot \Pr_{\bar{v}_i \sim \bar{D}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i};\bar{D})] \cdot \mathbf{I}_{v_i^* \leq \frac{16\mathrm{OPT}(\mathcal{I})}{\epsilon}} - \frac{\epsilon}{16} \cdot \mathrm{OPT}(\mathcal{I}) \\
& + \frac{1}{1+\frac{\epsilon}{4}} \sum_i \mathop{\mathbb{E}}_{\bar{v}_{-i} \sim \bar{D}_{-i}} p_i(\bar{v}_{-i};\bar{D}) \cdot \Pr_{\bar{v}_i \sim \bar{D}_i}[\bar{v}_i \geq p_i(\bar{v}_{-i};\bar{D})] \cdot \mathbf{I}_{v_i^* > \frac{16\mathrm{OPT}(\mathcal{I})}{\epsilon}} \\
=\ & \frac{1}{1+\frac{\epsilon}{4}} \mathrm{Rev}(\mathcal{M}_{MRS}(\bar{v},\bar{D})) - \frac{\epsilon}{16} \cdot \mathrm{OPT}(\mathcal{I}).
\end{aligned}
$$

By the optimality of Myerson's mechanism, $\mathrm{Rev}(\mathcal{M}_{MRS}(\bar{v},\bar{D})) = \mathrm{OPT}(\bar{\mathcal{I}})$. By Lemma 2 of [22], for $0 \leq \delta \leq 1$, $\mathrm{OPT}(\bar{\mathcal{I}}) \geq (1-\delta)\mathrm{OPT}(\mathcal{I})$ —that is, the optimal revenue under the discretized distribution is at least $(1-\delta)$ of the optimal revenue under the original distribution. Plugging in $\delta = \frac{\epsilon}{4}$, $\mathrm{Rev}(\mathcal{M}_{MRS}(\bar{v},\bar{D})) \geq (1-\frac{\epsilon}{4})\mathrm{OPT}(\mathcal{I})$. Thus we have

$$
\begin{aligned}
& \mathrm{Rev}(\mathcal{M}_{EMR}(\mathcal{I})) \\
\geq\ & \frac{1}{1+\frac{\epsilon}{4}}(1-\frac{\epsilon}{4})\mathrm{OPT}(\mathcal{I}) - \frac{\epsilon}{16} \cdot \mathrm{OPT}(\mathcal{I}) \geq \frac{1}{1+\epsilon}\mathrm{OPT}(\mathcal{I}).
\end{aligned}
$$

Thus Theorem 5 holds. □

Following [29], the sample complexity for single-item auction with regular distributions is $\tilde{\Theta}(n\epsilon^{-3})$. However, in the definition of sample complexity, each sample is a valuation profile of the players and consists of n values, and thus $\tilde{\Theta}(n^2\epsilon^{-3})$ values in total. When ϵ is small, the query complexity in this setting is $O(n\epsilon^{-1}\log\frac{n}{\epsilon})$, which is much lower than the sample complexity.

5.2. Lower bound

With regular distributions, by Lemma 3.6 in [23] it is sufficient to use a single sample (i.e., a random value drawn from the distribution) to achieve 2-approximation in revenue for *single-player* single-item auctions. Because every distribution is a uniform distribution in the quantile space, a sample for such auctions can be obtained by first choosing a quantile q uniformly at random from $[0, 1]$ and then making a quantile query. Thus, a single query is also sufficient for 2-approximation in this case. As such, unlike Theorem 4 where we have proved lower bounds for the query complexity for arbitrary constant approximations, for regular distributions we consider lower bounds for $(1+\epsilon)$-approximations, where ϵ is sufficiently small. More precisely, we have the following.

Theorem 6. *For any constant $\epsilon \in (0, \frac{1}{64})$, there exists a constant C such that, for any $n \geq 1$, any DSIC Bayesian mechanism $\mathcal{M}$ making less than $Cn\epsilon^{-1}$ non-adaptive value and quantile queries to the oracle, there exists a multi-player single-item Bayesian auction instance $\mathcal{I} = (N, M, D)$ where $|N| = n$ and D is regular, such that $\mathrm{Rev}(\mathcal{M}(\mathcal{I})) < \frac{\mathrm{OPT}(\mathcal{I})}{1+\epsilon}$.*

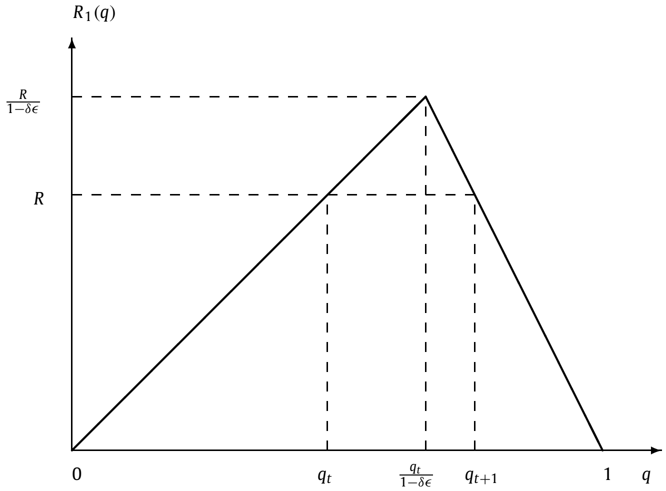

Fig. 1. The revenue curve of D_1.

We only prove Theorem 6 for the single-player case, as in the following lemma. The lower bound for general multi-player single-item auctions can be proved using the same technique as in Theorem 4, thus the full proof has been omitted.

Lemma 4. *For any constant $\epsilon \in (0, \frac{1}{64})$, there exists a constant C such that, for any DSIC Bayesian mechanism $\mathcal{M}$ making less than C/ϵ non-adaptive value and quantile queries to the oracle, there exists a single-player single-item Bayesian auction instance $\mathcal{I} = (N, M, D)$ where D is regular, such that* $\text{Rev}(\mathcal{M}(\mathcal{I})) < \frac{\text{OPT}(\mathcal{I})}{1+\epsilon}$.

Proof. Since the distributions are unbounded, we can always construct the distributions such that for any finite number of value queries, the responses for the value queries have almost none contribution to the optimal revenue. Thus we only need to focus on the lower bound for quantile queries.

Letting $k \triangleq \lceil \frac{1}{\delta\epsilon} \rceil$ and $C \triangleq \frac{1-2\delta\epsilon}{2\delta}$. Here δ is a constant to be determined later and δ, ϵ satisfies that $k \geq 2$. In our construction, we divide the quantile interval $[0, 1]$ into $k+1$ sub-intervals each, with the right-end points defined as follows: from left to right, $q_0 = 0$, $q_{t+1} = q_t + \delta\epsilon$ for each $t \in \{0, \dots, k-1\}$.

Accordingly, for any Bayesian mechanism $\mathcal{M}$ that makes less than $\frac{C}{\epsilon}$ non-adaptive quantile queries, there exists a quantile interval (q_t, q_{t+1}) such that, $q_{t+1} \leq 1 - 2\delta\epsilon$ and with probability at least $\frac{1}{2}$, no quantile in (q_t, q_{t+1}) is queried. Indeed, if this is not the case, then with probability at least $\frac{1}{2}$, all the quantile intervals except $(1-2\delta\epsilon, 1-\delta\epsilon)$ and $(1-\delta\epsilon, 1)$ are queried. Since there are at least $k-2$ quantile intervals, the expected total number of queries made by $\mathcal{M}$ is at least $\frac{k}{2} - 1 \geq \frac{1-2\delta\epsilon}{2\delta\epsilon} = \frac{C}{\epsilon}$, a contradiction.

We now construct two different single-player single-item Bayesian instances

$$\{\mathcal{I}_z = (N, M, D_z)\}_{z \in \{1,2\}},$$

where the distributions outside the quantile range (q_t, q_{t+1}) are all the same. Thus with probability at least $\frac{1}{2}$, mechanism $\mathcal{M}$ cannot distinguish the $\mathcal{I}_z$'s from each other. We then show that when this happens, mechanism $\mathcal{M}$ cannot be a $(1+3\epsilon)$-approximation for all instances $\mathcal{I}_z$.

Let R be a parameter that is large enough such that no value query will get any useful response. Then the first distribution D_1 with value bounded within $[0, \frac{R}{q_t}]$ is defined as follows, where $F_1(\cdot)$ is the cumulative probability function of D_1.

$$F_1(v) = \begin{cases} 1 - \frac{R}{(1-q_{t+1})v + R}, & 0 \leq v < \frac{R}{q_t}, \\ 1, & v = \frac{R}{q_t}. \end{cases}$$

That is there is a probability mass $\frac{q_t}{1-\delta\epsilon}$ at value $\frac{R}{q_t}$ and within interval $[0, \frac{R}{q_t})$ it is a continuous distribution. Then for any quantile in range $(0, \frac{q_t}{1-\delta\epsilon}]$, the oracle will response $\frac{R}{q_t}$. For quantile q in range $(\frac{q_t}{1-\delta\epsilon}, 1]$, the oracle will response $v(q) = \frac{R}{1-q_{t+1}}(\frac{1}{q} - 1)$. Therefore the revenue function with related to the quantile q is

$$R_1(q) = \begin{cases} \frac{R}{1-q_{t+1}}(1-q), & \frac{q_t}{1-\delta\epsilon} < q \leq 1, \\ \frac{R}{1-\delta\epsilon}, & q = \frac{q_t}{1-\delta\epsilon}. \end{cases}$$

The revenue curve $R_1(q)$ is illustrated in Fig. 1.

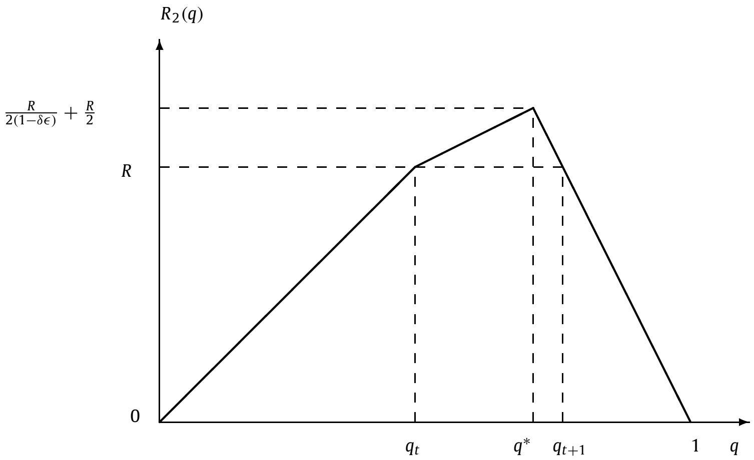

Fig. 2. The revenue curve of D_2.

The second distribution D_2 with value bounded within $[0, \frac{R}{q_t}]$ is defined as follows, where $F_2(\cdot)$ is the cumulative probability function of D_2. Let $v^* = \frac{R(2-\delta\epsilon)}{2(1-\delta\epsilon)-(2-\delta\epsilon)(1-q_{t+1})}$. Since $q_{t+1} \le 1-2\delta\epsilon$, $v^* > 0$ is well defined and it is easy to check $v^* < \frac{R}{q_t}$.

$$F_2(v) = \begin{cases} 1 - \frac{R}{(1-q_{t+1})v+R}, & 0 \le v < v^*, \\ 1 - \frac{R(1-\delta\epsilon)}{(1+q_t-\delta\epsilon)v-R}, & v^* \le v < \frac{R}{q_t}, \\ 1, & v = \frac{R}{q_t}. \end{cases}$$

That is, there is a probability mass q_t at value $\frac{R}{q_t}$ and a two-step continuous distribution within $[q_t, q^*]$ and $[q^*, q_{t+1}]$. Thus for any quantile in range $(0, q_t]$, the oracle will response $\frac{R}{q_t}$. It can be calculated that the quantile of value v^* is $q^* = 1 - \frac{2-\delta\epsilon}{2(1-\delta\epsilon)} \cdot (1-q_{t+1})$. Then for quantile q in range $(q_t, q^*]$, the oracle will response $v(q) = \frac{R}{q}(1 - \frac{q_t}{1+q_t-\delta\epsilon}) + \frac{R}{1+q_t-\delta\epsilon}$. For quantile q in range $(q^*, 1]$, the oracle will response $v(q) = \frac{R}{1-q_{t+1}}(\frac{1}{q} - 1)$. Therefore the revenue function with related to the quantile q is

$$R_2(q) = \begin{cases} \frac{R}{1-q_{t+1}}(1-q), & q^* < q \le 1, \\ \frac{R}{1+q_t-\delta\epsilon}(1+q-\delta\epsilon), & q_t \le q < q^*, \\ R, & q = q_t. \end{cases}$$

The revenue curve $R_2(q)$ is illustrated in Fig. 2.

Indeed when the quantile query is from $[0, q_t] \cup [q_{t+1}, 1]$, the oracle's answers for all distributions are the same. Accordingly, with probability at least $\frac{1}{2}$, mechanism $\mathcal{M}$ cannot distinguish D_z's from each other, which means it cannot distinguish $\mathcal{I}_z$'s from each other, as desired.

Since $\mathcal{M}$ is truthful, the allocation rule for the player must be monotone and he will pay the threshold payment set by $\mathcal{M}$, denoted by P. Let $P^* = \frac{(4-\delta\epsilon)R}{4(1-\delta\epsilon)-(4-\delta\epsilon)(1-q_{t+1})}$. Here P may be randomized. Recall that $\text{OPT}(\mathcal{I}_1) = \frac{R}{1-\delta\epsilon}$. If with probability $\frac{1}{2}$ setting the price $P \le P^*$, then for instance $\mathcal{I}_1$, we have

$$\begin{aligned} &\text{Rev}(\mathcal{M}(\mathcal{I}_1)) \le \frac{1}{2}\text{OPT}(\mathcal{I}_1) + \frac{1}{2}(\frac{3R}{4(1-\delta\epsilon)} + \frac{R}{4}) \\ = &\frac{7R}{8(1-\delta\epsilon)} + \frac{R}{8} = \frac{R}{1-\delta\epsilon}\left(1 - \frac{1}{8}\delta\epsilon\right) < \frac{\text{OPT}(\mathcal{I}_1)}{1+4\epsilon} \end{aligned}$$

when $\delta \ge 32$. On the other hand, recall that $\text{OPT}(\mathcal{I}_2) = \frac{R}{2(1-\delta\epsilon)} + \frac{R}{2} = \frac{(2-\delta\epsilon)R}{2(1-\delta\epsilon)}$. If with probability $\frac{1}{2}$, the price $P > P^*$, for instance $\mathcal{I}_2$, we have

$$\begin{aligned} &\text{Rev}(\mathcal{M}(\mathcal{I}_2)) < \frac{1}{2}\text{OPT}(\mathcal{I}_2) + \frac{(4-\delta\epsilon)R}{2(4-2\delta\epsilon)} = \frac{(2-\delta\epsilon)R}{4(1-\delta\epsilon)} + \frac{(4-\delta\epsilon)R}{2(4-2\delta\epsilon)} \\ = &\frac{(2-\delta\epsilon)R}{2(1-\delta\epsilon)}\left(\frac{1}{2} + \frac{(4-\delta\epsilon)(1-\delta\epsilon)}{2(2-\delta\epsilon)^2}\right) = \text{OPT}(\mathcal{I}_2)\left(1 - \frac{\delta\epsilon}{2(2-\delta\epsilon)^2}\right) < \frac{\text{OPT}(\mathcal{I}_2)}{1+4\epsilon} \end{aligned}$$

Mechanism 7 Mechanism $\mathcal{M}_{EVUD}$ for unit-demand auctions.

1: Given H and $\epsilon > 0$, run the value-query algorithm $\mathcal{A}_V$ with H and $\delta = \epsilon$ for each player i's distribution D_{ij} for each item j. Denote by D'_{ij} the returned distribution. Let $D'_i = \times_{j\in M} D'_{ij}$ and $D' = \times_{i\in N} D'_i$.
2: Run $\mathcal{M}_{UD}$ with D' and the players' reported values, $b = (b_{ij})_{i\in N, j\in M}$, to get allocation $x = (x_{ij})_{i\in N, j\in M}$ and price profile $p = (p_i)_{i\in N}$ as the outcome.

when $\delta \geq 32$. Thus for any mechanism $\mathcal{M}$ with $O(\frac{1}{\epsilon})$ quantile queries, there exists $z^* \in \{0, 1\}$ such that when $\epsilon < \frac{1}{64}$ and $\delta = 32$,

$$\text{Rev}(\mathcal{M}(\mathcal{I}_{z^*})) \leq \frac{\text{OPT}(\mathcal{I}_{z^*})}{2} + \frac{\text{OPT}(\mathcal{I}_{z^*})}{2(1+4\epsilon)} < \frac{\text{OPT}(\mathcal{I}_{z^*})}{1+\epsilon}.$$

Therefore Lemma 4 holds. □

6. Extension II: multi-item auctions

In this section, we generalize the query complexity to multi-item settings. For multi-item settings, the optimal BIC mechanism cannot be implemented in dominant strategies [45], and the goal in this section is to show that similar to single item settings, the simple DSIC mechanisms for unit-demand or additive settings can be implemented with limited queries. We will focus on the situation when there is no noise in the queries, and the case when the queries contain η-noise can be directly obtained in the same manner as in Section 3.3.

6.1. Bounded distributions

In this section, we consider multi-item auctions where all distributions are bounded within $[1, H]$, and we construct efficient query mechanisms whose query complexity matches our lower-bounds. We show that it is sufficient to use only value queries scheme $\mathcal{A}_V$ defined in Section 3.1.

The problem of unit-demand auctions and additive auctions is much more complicated compared to the single item auctions analyzed in Section 3. The optimal auction may involve lotteries and bundling, and the revenue monotonicity may not hold [33]. Even (disregarding complexity issues and) assuming we can design an optimal Bayesian mechanism for D', it is unclear how much revenue it guarantees under equilibrium when the players' values come from the true distribution D. To overcome this difficulty, we rely on recent developments on *simple* DSIC mechanisms with approximately optimal revenue.

The mechanism for unit-demand auctions is sequential post-price [36] and the analysis is relatively easy. For additive auctions, the Bayesian mechanism either runs Myerson's auction separately for each item or runs the VCG mechanism with a per-player entry fee [44,12]. However, an easy and direct analysis would lose a factor of m in the query complexity. To achieve a tight upper-bound, we need to really open the box and analyze the mechanism differently in several crucial places, exploring its behavior under oracle queries.

To sum up, given our query scheme, our mechanisms are black-box reductions to simple Bayesian mechanisms, thus are simple, natural, and easy to implement in practice, while the analysis is non-black-box, non-trivial and reveals interesting connections between Bayesian mechanisms and query schemes.

6.1.1. Unit-demand auctions

The main difficulty for unit-demand auctions is that we no longer have revenue monotonicity as in single-item auctions. Our analysis then comes in a non-blackbox way and relies on the *COPIES setting* [15,36], which provides an upper-bound for the optimal BIC revenue. By properly upper-bounding the optimal revenue in the COPIES setting under D', we are able to upper-bound the optimal revenue in unit-demand auctions using the expected revenue of $\mathcal{M}_{EVUD}$. More precisely, we have the following theorem.

Theorem 7. *$\forall\epsilon > 0$, for any unit-demand instance $\mathcal{I} = (N, M, D)$ with values in $[1, H]$, mechanism $\mathcal{M}_{EVUD}$ is DSIC, has query complexity $O(mn \log_{1+\epsilon} H)$, and $\text{Rev}(\mathcal{M}_{EVUD}(\mathcal{I})) \geq \frac{\text{OPT}(\mathcal{I})}{24(1+\epsilon)}$.*[9]

We prove Theorem 7 in the appendix. Below we provide some intuitions on designing mechanism $\mathcal{M}_{EVUD}$. Let us first recall the *sequential post-price* mechanism $\mathcal{M}_{UD}$. This mechanism processes the players one by one according to an arbitrary order, computes a price for each player i based on remaining items, remaining players and the prior distribution, and lets i choose his utility-maximizing item (or choose none). The revenue of this mechanism is analyzed by reducing the unit-demand instance to the *COPIES* setting, which we introduce below.

For a unit-demand auction instance $\mathcal{I} = (N, M, D)$, the corresponding COPIES instance is denoted by $\mathcal{I}^{CP} = (N^{CP}, M^{CP}, D)$, where each player $i \in N$ has m copies and each item $j \in M$ has n copies, and player i's copy j is only interested in item j's

[9] Note that in order to get the bound in Table 1, by setting ϵ as a constant such as 0.1, the approximation to the optimal revenue is a constant, and the query complexity is $O(mn \log H)$.

Mechanism 8 Mechanism $\mathcal{M}_{EVBVCG}$ to approximate $\mathcal{M}_{BVCG}$ via value queries.

1: Given H and $\epsilon > 0$, run the value-query algorithm $\mathcal{A}_V$ with H and $\delta = \sqrt{\epsilon+1} - 1$ for each player i's distribution D_{ij} for each item j. Denote by D'_{ij} the returned distribution. Let $D'_i = \times_{j \in M} D'_{ij}$ and $D' = \times_{i \in N} D'_i$.
2: Run $\mathcal{M}_{BVCG}$ with D' and the players' reported values, $b = (b_{ij})_{i \in N, j \in M}$, to get allocation $x = (x_{ij})_{i \in N, j \in M}$ and price profile $p = (p_i)_{i \in N}$ as the outcome.

copy i, with value v_{ij} drawn independently from D_{ij}. Thus $N^{CP} = M^{CP} = N \times M$, and $\mathcal{I}^{CP}$ is a single-parameter instance. Denote by N_i the set of player i's copies and by M_j the set of item j's copies. Note that both $\{N_i\}_{i \in N}$ and $\{M_j\}_{j \in M}$ are partitions of N^{CP} (and M^{CP}). Two natural constraints are imposed on feasible allocations under the COPIES setting, so as to connect it with the original unit-demand setting: (1) for each player i, at most one of his copies gets an item; and (2) for each item j, at most one of its copies gets allocated. Accordingly, letting q_s be the probability that a feasible mechanism allocates an item to a player copy $s \in N^{CP}$, we have $\sum_{s \in N_i} q_s \leq 1$ for each $i \in N$ and $\sum_{s \in M_j} q_s \leq 1$ for each $j \in M$.

The corresponding mechanism $\mathcal{M}_{UD}^{CP}$ for the COPIES setting works in the same way as $\mathcal{M}_{UD}$, except that it considers an arbitrary order of the players in N^{CP}, thus different copies of the same player may not be processed together. When evaluating the performance of mechanism $\mathcal{M}_{UD}^{CP}$, the order of the players is chosen by an *online adaptive adversary,* who tries to minimize the expected revenue of the mechanism. Because this adversary is the worst-case for mechanism $\mathcal{M}_{UD}^{CP}$,

$$\text{Rev}(\mathcal{M}_{UD}(\mathcal{I}; D')) \geq \text{Rev}(\mathcal{M}_{UD}^{CP}(\mathcal{I}^{CP}; D'))$$

for any distribution D', where the latter is the expected revenue of $\mathcal{M}_{UD}^{CP}$ under the online adaptive adversary. Indeed, mechanism $\mathcal{M}_{UD}$ can be considered as $\mathcal{M}_{UD}^{CP}$ under a specific order where all copies of each player come together, thus the revenue is at least that when the order of N^{CP} is adaptively chosen by the adversary. Given the above discussion, we are able to prove Theorem 7 and the formal proof is in Appendix B.

6.1.2. Additive auctions

For additive auctions, the DSIC Bayesian mechanism in [44,12] chooses between two mechanisms, whichever generates higher expected revenue under the true prior D. The first is the “individual Myerson” mechanism, denoted by $\mathcal{M}_{IM}$, which sells each item separately using Myerson's mechanism. The second is the VCG mechanism with optimal per-player entry fees, denoted by $\mathcal{M}_{BVCG}$.

In our mechanism $\mathcal{M}_{EVA}$, the seller queries about D using algorithm $\mathcal{A}_V$ with properly chosen parameters. Given the resulting distribution D', the seller either runs $\mathcal{M}_{IM}$ or runs $\mathcal{M}_{BVCG}$ as a blackbox, resulting in query mechanisms $\mathcal{M}_{EVIM}$ and $\mathcal{M}_{EVBVCG}$. We only define the latter in Mechanism 8, and the former simply replaces $\mathcal{M}_{BVCG}$ with $\mathcal{M}_{IM}$. Note that $\text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I})) = \text{Rev}(\mathcal{M}_{IM}(\mathcal{I}; D'))$ and $\text{Rev}(\mathcal{M}_{EVBVCG}(\mathcal{I})) = \text{Rev}(\mathcal{M}_{BVCG}(\mathcal{I}; D'))$. However, the seller cannot compute these two revenue and choose the better one, because he does not know D. Thus he randomly chooses between the two, according to probabilities defined in our analysis, to optimize the approximation ratio. We have the following theorem.

Theorem 8 is harder to show. Indeed, one cannot use revenue monotonicity or the COPIES setting to easily upper-bound the optimal BIC revenue. Our analysis is based on the duality framework of [12] for Bayesian auctions, properly adapted for the query setting.

Theorem 8. $\forall \epsilon > 0$, *for any additive instance* $\mathcal{I} = (N, M, D)$ *with values in* $[1, H]$, *mechanism* $\mathcal{M}_{EVA}$ *is DSIC, has query complexity* $O(mn \log_{1+\epsilon} H)$, *and* $\text{Rev}(\mathcal{M}_{EVA}(\mathcal{I})) \geq \frac{\text{OPT}(\mathcal{I})}{8(1+\epsilon)}$.

The proof of Theorem 8 required unraveling the details from the duality framework and the core-tail decomposition technique introduced in [12,44]. Intuitively, selling items separately covers the revenue contribution of the optimal mechanism from the tail and selling items with entry fees covers the revenue contribution from the core. We show that given access to value queries, those two mechanisms still separately covers the revenue contribution of the optimal mechanism from the tail and the core with original distributions. Thus randomly choosing from $\mathcal{M}_{EVIM}$ and $\mathcal{M}_{EVBVCG}$ provides an $8(1+\epsilon)$-approximation to the optimal revenue. The details of the proofs resemble the techniques in [12], which is relegated to Appendix B.

6.2. Unbounded distributions

In this section, we consider unbounded distributions but have small tails. We provide upper bounds on the number of quantile queries required for implementing the approximately revenue optimal mechanisms.

6.2.1. Unit-demand auctions

The unit-demand mechanism $\mathcal{M}_{EQUD}$ is similar (see Mechanism 9), and we have the following.

Theorem 9. $\forall \epsilon > 0$, *any unit-demand instance* $\mathcal{I} = (N, M, D)$ *satisfying Small-Tail Assumption,* $\mathcal{M}_{EQUD}$ *is DSIC, has query complexity* $O(-mn \log_{1+\frac{\epsilon}{3}} h(\frac{2\epsilon}{3(1+\epsilon)}))$, *and* $\text{Rev}(\mathcal{M}_{EQUD}(\mathcal{I})) \geq \frac{\text{OPT}(\mathcal{I})}{24(1+\epsilon)}$.

Mechanism 9 Mechanism $\mathcal{M}_{EQUD}$ for unit-demand auctions.

1: Given $\epsilon > 0$, run algorithm $\mathcal{A}_Q$ with $\delta = \frac{\epsilon}{3}$ and $\epsilon_1 = h(\frac{2\epsilon}{3(1+\epsilon)})$ (i.e., $\delta_1 = \frac{2\epsilon}{3(1+\epsilon)}$ for Small Tail Assumption 2), for each player i's distribution D_{ij} on each item j. Denote by D'_{ij} the returned distribution. Let $D'_i = \times_{j\in M} D'_{ij}$ and $D' = \times_{i\in N} D'_i$.
2: Run mechanism $\mathcal{M}_{UD}$ with D' and the players' reported values, $b = (b_{ij})_{i\in N, j\in M}$, to get allocation $x = (x_{ij})_{i\in N, j\in M}$ and price profile $p = (p_i)_{i\in N}$ as the outcome.

Mechanism 10 Mechanism $\mathcal{M}_{EQBVCG}$ for additive auctions.

1: Given $\epsilon > 0$, run algorithm $\mathcal{A}_Q$ with $\delta = (1+\frac{\epsilon}{5})^{1/m} - 1$ and $\epsilon_1 = h(\frac{\epsilon}{10(1+\epsilon)})$ (i.e., $\delta_1 = \frac{\epsilon}{10(1+\epsilon)}$ for Small Tail Assumption 1), for each player i's distribution D_{ij} on each item j. Denote by D'_{ij} the returned distribution. Let $D'_i = \times_{j\in M} D'_{ij}$ and $D' = \times_{i\in N} D'_i$.
2: Run $\mathcal{M}_{BVCG}$ with D' and the players' reported values, $b = (b_{ij})_{i\in N, j\in M}$, to get allocation $x = (x_{ij})_{i\in N, j\in M}$ and price profile $p = (p_i)_{i\in N}$ as the outcome.

The proof of Theorem 9 is similar to that of Theorem 7, but Lemma 2 above is used instead of Lemma 5 of [22], and the round-down scheme is replaced by the randomized round-down scheme designed in the proof of Lemma 2. The details have been omitted.

6.2.2. Additive auctions

For additive auctions, to approximate $\mathcal{M}_{BVCG}$, not only we need the Small-Tail Assumption, but we also approximate D by running the quantile-query algorithm $\mathcal{A}_Q$ with different parameters. The resulting mechanism $\mathcal{M}_{EQBVCG}$ is defined in Mechanism 10, and the mechanism $\mathcal{M}_{EQIM}$ simply replaces $\mathcal{M}_{BVCG}$ with $\mathcal{M}_{IM}$. Again, in the final mechanism $\mathcal{M}_{EQA}$ the seller randomly chooses between the two query mechanisms, according to probabilities defined in the analysis. We have the following theorem.

Theorem 10. $\forall\epsilon > 0$, *any additive instance* $\mathcal{I} = (N, M, D)$ *satisfying Small-Tail Assumption,* $\mathcal{M}_{EQA}$ *is DSIC, has query complexity* $O(-m^2 n \log_{1+\frac{\epsilon}{5}} h(\frac{\epsilon}{10(1+\epsilon)}))$, *and* $\mathrm{Rev}(\mathcal{M}_{EQA}(\mathcal{I})) \geq \frac{\mathrm{OPT}(\mathcal{I})}{8(1+\epsilon)}$.

The proof of Theorem 10 is technical which requires a deep understanding of mechanism $\mathcal{M}_{EQBVCG}$, and thus we defer the proof to Appendix B as well.

The main advantage of using quantile queries is to handle unbounded distributions. In addition, we can use the resulting query mechanisms to construct *sampling* mechanisms; see Section 7. As shown in Theorem 10, the query complexity of mechanism $\mathcal{M}_{EQA}$ has an extra factor of m compared with that of $\mathcal{M}_{EVA}$ (and the lower bound). It would be interesting to see whether our lower-bounds can be improved in this scenario.

6.3. Using quantile queries for bounded distributions

As a corollary, Theorems 2, 9 and 10 also provide another way to approximate the optimal BIC revenue using only quantile queries when the distributions are bounded. More precisely, we have the following.

Corollary 1. *For any* $\epsilon > 0$, $H > 1$, *and prior distribution* D *with each* D_{ij} *bounded within* $[1, H]$, *there exist DSIC mechanisms that use* $O(mn \log_{1+\epsilon} \frac{nmH(1+\epsilon)}{\epsilon})$ *quantile queries for single-item auctions and unit-demand auctions, and use* $O(m^2 n \log_{1+\epsilon} \frac{nmH(1+\epsilon)}{\epsilon})$ *quantile queries for additive auctions, whose approximation ratios to* OPT *are respectively* $1+\epsilon$, $24(1+\epsilon)$ *and* $8(1+\epsilon)$.

Proof. We only need to show that the Small-Tail Assumption is naturally satisfied when the distributions have bounded supports. For example, consider additive auctions where all values are in $[1, H]$, as considered in [35,18]. Then mH and 1 are straightforward upper- and lower-bounds for $\mathrm{OPT}(\mathcal{I})$, respectively. Moreover, by individual rationality, mH is an upper-bound for the revenue generated under any valuation profiles. Given δ_1, let $\epsilon_1 = h(\delta_1) = \frac{\delta_1}{m^2 nH}$ and denote by E the event that there exist at least one player i and one item j with $q_{ij}(v_{ij}) \leq \epsilon_1$. By the union bound, $\Pr[E] \leq mn\epsilon_1 = mn \cdot \frac{\delta_1}{m^2 nH} = \frac{\delta_1}{mH}$. Therefore

$$\mathop{\mathbb{E}}_{v\sim D} \mathbf{I}_{\exists i,j,q_{ij}(v_{ij})\leq\epsilon_1} \mathrm{Rev}(\mathcal{M}(v;\mathcal{I})) \leq mH \cdot \Pr[E] \leq \delta_1 \leq \delta_1 \mathrm{OPT}(\mathcal{I}).$$

Combining this observation with Theorems 2, 9 and 10, we have Corollary 1 when the values are all bounded in $[1, H]$. □

Remark: Since bounded distributions are special cases of unbounded distributions with the Small-tail Assumption. The lower bound for bounded distributions in Theorem 4 can be directly applied to obtain the lower bounds for unbounded distributions with the Small-tail Assumption.

Mechanism 11 Sampling mechanism $\mathcal{M}_{SM}$.

1: For single-item auctions and unit-demand auctions, given $\epsilon > 0$, set $\delta = \frac{\epsilon}{6}$, $\epsilon_1 = h(\frac{2\epsilon}{3(1+\epsilon)})$ and $k = \lceil \log_{1+\delta} \frac{1}{\epsilon_1} \rceil$; define the *quantile vector* as $q = (q_0, q_1, \ldots, q_{k-1}, q_k) = (1, \epsilon_1(1+\delta)^{k-1}, \ldots, \epsilon_1(1+\delta), \epsilon_1)$.
For additive auctions, given $\epsilon > 0$, set $\epsilon_1 = h(\frac{\epsilon}{10(1+\epsilon)})$ and $k = \lfloor \frac{1}{\epsilon_1} \rfloor$; define the *quantile vector* as $q = (q_0, q_1, \ldots, q_{k-1}, q_k) = (1, k\epsilon_1, \ldots, 2\epsilon_1, \epsilon_1)$.
2: For each player i and item j, given t samples $V_{ij}^t = \{v_{ij}^1, \ldots, v_{ij}^t\}$, without loss of generality assume $v_{ij}^1 \geq v_{ij}^2 \geq \cdots \geq v_{ij}^t$. For each quantile q_l, set $v_{ij}^{tq_l}$ to be the value corresponding to the quantile query q_l. (If tq_l is not an integer then the mechanism takes $\lceil tq_l \rceil$.)
3: Construct a discrete distribution D'_{ij} as follows: $D'_{ij}(v_{ij}^{tq_l}) = q_l - q_{l+1}$ for each $l \in \{0, \ldots, k-1\}$, and $D'_{ij}(v_{ij}^{tq_k}) = \epsilon_1$. Finally, let $D'_i = \times_{j \in M} D'_{ij}$ for each player i and let $D' = \times_{i \in N} D'_i$.
4: Run $\mathcal{M}_{MRS}/\mathcal{M}_{UD}/\mathcal{M}_A$ with distribution D' and the players' reported values.

7. Extension III: sampling mechanisms

Using our techniques for query complexity, we can easily construct sampling mechanisms for multi-parameter auctions. Currently, the sample complexity for unit-demand auctions and additive distributions has been upper-bounded in [4,38,26,9] for bounded auctions. In this section, we provide another way to explicitly construct sampling mechanisms for both unit-demand and additive auctions, for arbitrary distributions with small-tails as well as for bounded distributions. As we will see, the revenue approximation ratios obtained in this section for unit-demand and for additive auctions are better than the results shown in Table 2.

The idea is to use samples to approximate *quantile* queries. Mechanism 11 defines our sampling mechanism $\mathcal{M}_{SM}$. Recall that mechanisms $\mathcal{M}_{MRS}$, $\mathcal{M}_{UD}$ and $\mathcal{M}_A$ are known (approximately) optimal DSIC mechanisms for single-item, unit-demand and additive auctions respectively. Note that in mechanism $\mathcal{M}_{SM}$, we use a different method to discretize the quantile space for additive auctions, so as to further reduce its sample complexity. In particular, we have the following theorem, which is proved in Appendix C.

Theorem 11. *$\forall \epsilon > 0$ and $\gamma \in (0, 1)$, for any Bayesian instance $\mathcal{I} = (N, M, D)$,*

- *for single-item auctions satisfying the Small-Tail Assumption, with $\tilde{O}(h^{-2}(\frac{2\epsilon}{3(1+\epsilon)}) \cdot (\frac{\epsilon}{1+\epsilon})^{-2})$ samples, mechanism $\mathcal{M}_{SM}$ achieves revenue at least $\frac{1}{1+\epsilon}\mathrm{OPT}(\mathcal{I})$ with probability at least $1-\gamma$;*
- *for unit-demand auctions satisfying the Small-Tail Assumption, with $\tilde{O}(h^{-2}(\frac{2\epsilon}{3(1+\epsilon)}) \cdot (\frac{\epsilon}{1+\epsilon})^{-2})$ samples, mechanism $\mathcal{M}_{SM}$ achieves revenue at least $\frac{1}{24(1+\epsilon)}\mathrm{OPT}$ with probability at least $1-\gamma$;*
- *for additive auctions satisfying the Small-Tail Assumption, with $\tilde{O}(h^{-2}(\frac{\epsilon}{10(1+\epsilon)})(\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}})^{-2})$ samples, mechanism $\mathcal{M}_{SM}$ achieves revenue at least $\frac{1}{8(1+\epsilon)}\mathrm{OPT}$ with probability at least $1-\gamma$.*

Remark Following the convention in the literature, a logarithmic factor depending on γ has been absorbed in $\tilde{O}(\cdot)$. If the values are bounded in $[1, H]$, by defining the tail function h according to H, the resulting sample complexity is $\tilde{O}(m^4n^2H^2(1+\epsilon)^4\epsilon^{-4})$ for unit-demand auctions and $\tilde{O}(m^4n^2H^2(\frac{1+\epsilon}{\epsilon})^2(\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}})^{-2})$ for additive auctions.

8. Conclusion and future directions

We studied the query complexity of Bayesian mechanisms in this work, where the seller only has limited oracle accesses to the players' distributions, via quantile and value queries. We prove logarithmic lower bounds for any constant approximation DSIC mechanisms for single-item auctions and multi-item auctions with subadditive valuations. For single-item, unit-demand and additive auctions, we prove tight upper-bounds via efficient query schemes. Thus, in those settings the seller needs to access much less than the entire distribution to achieve approximately optimal revenue. As this is the first time the query complexity of Bayesian auctions is considered, many interesting questions about the query complexity of Bayesian auctions are worth exploring. First, as mentioned, there is a logarithmic gap between adaptive queries and non-adaptive queries for multi-item auctions. It is intriguing to design approximately optimal Bayesian mechanisms with matching query complexity using adaptive queries. Moreover, in this work, we have focused on designing query mechanisms for additive and unit-demand valuations, another interesting direction is to design query mechanisms for general subadditive valuations. Finally, it is interesting to study more complicated settings when the oracles are also rational players or even the buyers themselves.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Missing proofs for Section 3

We now prove the claims used above.

Claim 1 (restated). $\Pr_{v_i \sim D_i}[v_i \geq p_i(v_{-i}; D) | q_i(v_i) > \epsilon_1] \leq (1+\delta) \Pr_{v_i \sim D_i}[v_i^- \geq p_i(v_{-i}; D)]$.

Proof. By definition, $q_i(v_i) > \epsilon_1$ implies $v_i \leq v'_{i;k}$, where $v'_{i;k}$ is the largest value in V'_i, the support of distribution D'_i. Note that $v_i^- \leq v'_{i;k}$ for any v_i. If $p_i(v_{-i}; D) > v'_{i;k}$, then both probabilities are 0 and the inequality holds.

Below we consider the case $p_i(v_{-i}; D) \leq v'_{i;k}$. Let $v'_{i;-1} = -1$ and $l \in \{0, 1, \ldots, k\}$ be such that $v'_{i;l} \geq p_i(v_{-i}; D)$ and $v'_{i;l-1} < p_i(v_{-i}; D)$. We have

$$
\begin{aligned}
& \Pr_{v_i \sim D_i}[v_i \geq p_i(v_{-i}; D) | q_i(v_i) > \epsilon_1] \\
\leq & \Pr_{v_i \sim D_i}[v_i^+ \geq p_i(v_{-i}; D) | q_i(v_i) > \epsilon_1] \\
= & \Pr_{v_i \sim D_i}[v_i^+ \geq v'_{i;l} | q_i(v_i) > \epsilon_1] \\
= & \Pr_{v_i \sim D_i}[v_i^- \geq v'_{i;l-1} | q_i(v_i) > \epsilon_1] \\
= & \Pr_{v_i \sim D_i}[v_i^- \geq v'_{i;\max\{0,l-1\}} | q_i(v_i) > \epsilon_1] \\
= & \frac{\Pr_{v_i \sim D_i}[v_i^- \geq v'_{i;\max\{0,l-1\}} \text{ and } q_i(v_i) > \epsilon_1]}{\Pr_{v_i \sim D_i}[q_i(v_i) > \epsilon_1]} \\
= & \frac{\Pr_{v_i \sim D_i}[v_i^- \geq v'_{i;\max\{0,l-1\}}] - \Pr_{v_i \sim D_i}[v_i^- \geq v'_{i;\max\{0,l-1\}} \text{ and } q_i(v_i) \leq \epsilon_1]}{\Pr_{v_i \sim D_i}[q_i(v_i) > \epsilon_1]} \\
= & \frac{\Pr_{v_i \sim D_i}[v_i^- \geq v'_{i;\max\{0,l-1\}}] - \Pr_{v_i \sim D_i}[q_i(v_i) \leq \epsilon_1]}{\Pr_{v_i \sim D_i}[q_i(v_i) > \epsilon_1]} \\
\leq & \Pr_{v_i \sim D_i}[v_i^- \geq v'_{i;\max\{0,l-1\}}] \\
= & \Pr_{v'_i \sim D'_i}[v'_i \geq v'_{i;\max\{0,l-1\}}] = q_{\max\{0,l-1\}} \leq (1+\delta) q_l \\
= & (1+\delta) \Pr_{v'_i \sim D'_i}[v'_i \geq v'_{i;l}] = (1+\delta) \Pr_{v'_i \sim D'_i}[v'_i \geq p_i(v_{-i}; D)] \\
= & (1+\delta) \Pr_{v_i \sim D_i}[v_i^- \geq p_i(v_{-i}; D)],
\end{aligned}
$$

as desired. Indeed, the first inequality is because $v_i^+ > v_i$, and the first equality is because $v_i^+ \in V'_i \cup \{+\infty\}$ and thus $v_i^+ \geq p_i(v_{-i}; D)$ if and only if $v_i^+ \geq v'_{i;l}$. Similarly, the second equality is because (v_i^-, v_i^+) and $(v'_{i;l-1}, v'_{i;l})$ are two pairs of consecutive values in $V'_i \cup \{-1, +\infty\}$, thus $v_i^+ \geq v'_{i;l}$ if and only if $v_i^- \geq v'_{i;l-1}$. The third equality is because $v_i^- \geq v'_{i;0}$ always. The sixth equality is because $q_i(v_i) \leq \epsilon_1$ implies $v_i \geq v'_{i;k} \geq v'_{i;l}$, thus $v_i^- \geq v'_{i;\max\{0,l-1\}}$. The seventh equality is by the definition of the round-down scheme. The following two equalities and the inequality are by the construction of D'_i and the definition of the quantile vector q. Indeed, $(1+\delta)q_0 = 1+\delta > 1 = q_0$, $(1+\delta)q_1 = \epsilon_1(1+\delta)^k \geq \epsilon_1(1+\delta)^{\log_{1+\delta}\frac{1}{\epsilon_1}} = \epsilon_1 \cdot \frac{1}{\epsilon_1} = 1 = q_0$, and $(1+\delta)q_l = q_{l-1}$ for any $l \geq 2$. The second-last equality is because $v'_i \in V'_i$, thus $v'_i \geq v'_{i;l}$ if and only if $v'_i \geq p_i(v_{-i}; D)$. Finally, the last equality is again by the definition of the round-down scheme. □

Claim 2 (restated). $\text{Rev}(\mathcal{M}^*(\mathcal{I}')) \geq \frac{1}{1+\epsilon}\text{OPT}(\mathcal{I})$.

Proof. Combining Equation (2) and Claim 1, we have

$$
\text{Rev}(\mathcal{M}^*(\mathcal{I}')) \geq \frac{1}{1+\delta} \sum_i \mathbb{E}_{v_{-i} \sim D_{-i}} p_i(v_{-i}; D) \cdot \Pr_{v_i \sim D_i}[v_i \geq p_i(v_{-i}; D) | q_i(v_i) > \epsilon_1].
$$

Accordingly,

$$
\text{Rev}(\mathcal{M}^*(\mathcal{I}')) \geq \frac{1}{1+\delta} \sum_i \mathbb{E}_{v_{-i} \sim D_{-i}} p_i(v_{-i}; D) \cdot \Pr_{v_i \sim D_i}[v_i \geq p_i(v_{-i}; D) | q_i(v_i) > \epsilon_1]
$$

$$
\begin{aligned}
&\geq \frac{1}{1+\delta}\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} p_i(v_{-i};D)\cdot \Pr_{v_i\sim D_i}[q_i(v_i)>\epsilon_1 \text{ and } v_i \geq p_i(v_{-i};D)]\\
&= \frac{1}{1+\delta}\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i\sim D_i} p_i(v_{-i};D)\cdot \mathbf{I}_{q_i(v_i)>\epsilon_1}\cdot \mathbf{I}_{v_i\geq p_i(v_{-i};D)}\\
&= \frac{1}{1+\delta}\mathop{\mathbb{E}}_{v\sim D}\sum_i p_i(v_{-i};D)\cdot \mathbf{I}_{q_i(v_i)>\epsilon_1}\cdot \mathbf{I}_{v_i\geq p_i(v_{-i};D)}\\
&\geq \frac{1}{1+\delta}\mathop{\mathbb{E}}_{v\sim D}\mathbf{I}_{\forall i, q_i(v_i)>\epsilon_1}\cdot \sum_i p_i(v_{-i};D)\mathbf{I}_{v_i\geq p_i(v_{-i};D)}\\
&= \frac{1}{1+\delta}\mathop{\mathbb{E}}_{v\sim D}\mathbf{I}_{\forall i, q_i(v_i)>\epsilon_1}\cdot \mathrm{Rev}_{\mathrm{OPT}}(v;\mathcal{I})\\
&\geq \frac{1-\delta_1}{1+\delta}\mathrm{OPT}(\mathcal{I}). \qquad (11)
\end{aligned}
$$

Here the second last equality holds by the definition of $p_i(v_{-i}; D)$ and $\mathrm{Rev}_{\mathrm{OPT}}(v;\mathcal{I})$, and last inequality holds by the Small-Tail Assumption. Since $\delta = \frac{\epsilon}{3}$ and $\delta_1 = \frac{2\epsilon}{3(1+\epsilon)}$, we have

$$
\frac{1-\delta_1}{1+\delta} = \frac{1}{1+\epsilon},
$$

thus Claim 2 holds. □

Appendix B. Missing proofs for Section 6

Theorem 7 (restated). *$\forall\epsilon > 0$, for any unit-demand instance $\mathcal{I} = (N, M, D)$ with values in $[1, H]$, mechanism $\mathcal{M}_{EVUD}$ is DSIC, has query complexity $O(mn\log_{1+\epsilon} H)$, and $\mathrm{Rev}(\mathcal{M}_{EVUD}(\mathcal{I})) \geq \frac{\mathrm{OPT}(\mathcal{I})}{24(1+\epsilon)}$.*

Proof. It is easy to see that the query complexity of $\mathcal{M}_{EVUD}$ is $O(mn\log_{1+\epsilon} H)$, since each distribution D_{ij} needs $O(\log_{1+\epsilon} H)$ value queries. Also, it is immediate that $\mathcal{M}_{EVUD}$ is DSIC.

Below we prove the revenue bound. By construction,

$$
\mathrm{Rev}(\mathcal{M}_{EVUD}(\mathcal{I})) = \mathrm{Rev}(\mathcal{M}_{UD}(\mathcal{I}; D')). \qquad (12)
$$

Let $\mathcal{I}' = (N, M, D')$ and $\mathcal{I}'^{CP} = (N^{CP}, M^{CP}, D')$. We state the following lemma, which is proved later. Intuitively, the lemma states the fact that in mechanism $\mathcal{M}^{CP}_{UD}$, the revenue of the seller is not affected when the true valuation distributions of the players are shifted from D' to D.

Lemma 5. $\mathrm{Rev}(\mathcal{M}_{UD}(\mathcal{I}; D')) \geq \mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}^{CP}; D')) = \mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}'^{CP}))$.

By Theorem 1 of [36], the sequential post-price mechanism is at least a 6-approximation to the optimal BIC revenue in the COPIES setting. Thus

$$
\mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}'^{CP})) \geq \frac{1}{6}\mathrm{OPT}(\mathcal{I}'^{CP}). \qquad (13)
$$

Next, because the COPIES setting is a single-parameter setting, and because of the way we discretize the value space in algorithm $\mathcal{A}_V$, by Lemma 5 of [22] we have

$$
\mathrm{OPT}(\mathcal{I}'^{CP}) \geq \frac{1}{1+\epsilon}\mathrm{OPT}(\mathcal{I}^{CP}). \qquad (14)
$$

Finally, by Theorem 6 of [12], the optimal BIC revenue in the COPIES setting is a 4-approximation to the optimal BIC revenue in the original unit-demand setting. Thus

$$
\mathrm{OPT}(\mathcal{I}^{CP}) \geq \frac{1}{4}\mathrm{OPT}(\mathcal{I}). \qquad (15)
$$

Combining Equations (12), (13), (14), (15) and Lemma 5, Theorem 7 holds. □

It remains to prove Lemma 5.

Proof of Lemma 5. The inequality is already explained in Section 6.1.1. Now we prove the equality. For any value profile $v \sim D$, let v' be v rounded down to the support of D'. That is, for each v_{ij}, v'_{ij} is the largest value in the support of D'_{ij} that is less than or equal to v_{ij}. Recall that the support of D'_{ij} is the set $\{v_0, \cdots, v_k\}$ as defined in the query algorithm $\mathcal{A}_V$. By the definition of D'_{ij}, for any $0 \le l \le k-1$,

$$\Pr_{v_{ij}\sim D_{ij}}[v'_{ij} = v_l] = \Pr_{v_{ij}\sim D_{ij}}[v_{ij} \ge v_l] - \Pr_{v_{ij}\sim D_{ij}}[v_{ij} \ge v_{l+1}] = q(v_l) - q(v_{l+1}) = q_l - q_{l+1} = D'_{ij}(v_l),$$

and

$$\Pr_{v_{ij}\sim D_{ij}}[v'_{ij} = v_k] = \Pr_{v_{ij}\sim D_{ij}}[v_{ij} \ge v_k] = q(v_k) = q_k = D'_{ij}(v_k).$$

That is, if v is distributed according to D then v' is distributed according to D'.

For any value profile v and the corresponding v', arbitrarily fix an order σ of the players in N^{CP}, which is a bijection from $\{1, \cdots, mn\}$ to $\{1, \cdots, mn\}$. Without loss of generality, each player $\sigma(s)$ gets the corresponding item $\sigma(s)$ whenever his true value is greater than or equal to the posted price for him. Below we show that mechanism $\mathcal{M}^{CP}_{UD}$ produces the same outcome no matter the players' true values are v or v'. That is, for any $s \in \{1, \ldots, mn\}$, (1) $\mathcal{M}^{CP}_{UD}$ produces the same price $p_{\sigma(s)}$ under v and v' for player $\sigma(s)$, and (2) $v_{\sigma(s)} \ge p_{\sigma(s)}$ if and only if $v'_{\sigma(s)} \ge p_{\sigma(s)}$.

To prove these two properties, note that by the construction of mechanism $\mathcal{M}^{CP}_{UD}$, the price $p_{\sigma(s)}$ posted to $\sigma(s)$ depends only on the distribution D' and the set $A_{\sigma(s)}$ of items sold to the players arriving before $\sigma(s)$. Here $p_{\sigma(s)}$ may be randomized if $D'_{\sigma(s)}$ is irregular, but it always takes value in the support of $D'_{\sigma(s)}$ (except that, if selling the corresponding item $\sigma(s)$ to player $\sigma(s)$ is not feasible anymore, then $p_{\sigma(s)} = +\infty$).

We prove the two desired properties by induction. When $s = 1$, property (1) trivially holds, because $A_{\sigma(1)} = \emptyset$ under both value profiles. Furthermore, because a realization of $p_{\sigma(1)}$ is always in the support of $D'_{\sigma(1)}$, and because $v'_{\sigma(1)}$ is $v_{\sigma(1)}$ rounded down to the support of $D'_{\sigma(1)}$, property (2) holds when $s = 1$.

Now assume (1) and (2) hold for any $s \le t$ with $t < mn$. We show they also hold for $s = t + 1$. Indeed, the inductive hypothesis implies that for any $s \le t$, $A_{\sigma(s)}$ is the same under the two value profiles. In particular, $A_{\sigma(t+1)}$ is the same, which means the price $p_{\sigma(t+1)}$ is the same. Thus property (1) holds. Property (2) also holds because a realization of $p_{\sigma(t+1)}$ is always in the support of $D'_{\sigma(t+1)}$. In sum, for any order σ, mechanism $\mathcal{M}^{CP}_{UD}$ produces the same outcome under the two value profiles v and v', thus the same revenue.

Accordingly, under the online adaptive adversary for $(\mathcal{I}^{CP}; D')$, the revenue $\mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}^{CP}; D'))$ is the same as the revenue when the players' true values are obtained by rounding $v \sim D$ to v'. Because the resulting v' is distributed according to D', $\mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}^{CP}; D'))$ is at least the expected revenue of $\mathcal{M}^{CP}_{UD}$ under the online adaptive adversary for $\mathcal{I}'^{CP}$. Indeed, a randomized adversary for $\mathcal{I}'^{CP}$ can simulate the adversary for $(\mathcal{I}^{CP}; D')$: in each step, given v'_s with $s \in N^{CP}$ being the player in this step, the former first samples v_s from D_s conditional on v_s rounded down to v'_s, and then uses the latter to decide which player arrives next. Thus,

$$\mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}^{CP}; D')) \ge \mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}'^{CP})).$$

Similarly,

$$\mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}^{CP}; D')) \le \mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}'^{CP})).$$

Therefore $\mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}^{CP}; D')) = \mathrm{Rev}(\mathcal{M}^{CP}_{UD}(\mathcal{I}'^{CP}))$ and Lemma 5 holds. □

Theorem 8 (restated). *$\forall \epsilon > 0$, for any additive instance $\mathcal{I} = (N, M, D)$ with values in $[1, H]$, mechanism $\mathcal{M}_{EVA}$ is DSIC, has query complexity $O(mn \log_{1+\epsilon} H)$, and $\mathrm{Rev}(\mathcal{M}_{EVA}(\mathcal{I})) \ge \frac{\mathrm{OPT}(\mathcal{I})}{8(1+\epsilon)}$.*

Proof. First, it is easy to see that the query complexity of mechanism $\mathcal{M}_{EVA}$ is $O(mn \log_{1+\delta} H)$, since there are in total mn distributions and each one of them needs $O(\log_{1+\delta} H)$ value queries in the algorithm $\mathcal{A}_V$. Since $\delta = \sqrt{\epsilon + 1} - 1$, $O(mn \log_{1+\delta} H) = O(mn \log_{1+\epsilon} H)$. Second, since mechanisms $\mathcal{M}_{BVCG}$ and $\mathcal{M}_{IM}$ are both DSIC, $\mathcal{M}_{EVA}$ is DSIC.

Recall that mechanism $\mathcal{M}_{EVA}$ randomly chooses between running $\mathcal{M}_{EVIM}$ and running $\mathcal{M}_{EVBVCG}$. Therefore, to upper-bound the optimal revenue $\mathrm{OPT}(\mathcal{I})$ using $\mathrm{Rev}(\mathcal{M}_{EVA}(\mathcal{I}))$, we only need to upper-bound $\mathrm{OPT}(\mathcal{I})$ using $\mathrm{Rev}(\mathcal{M}_{EVIM}(\mathcal{I}))$ and $\mathrm{Rev}(\mathcal{M}_{EVBVCG}(\mathcal{I}))$.

As in [12], we only need to consider the prior distribution D with finite support. Let V_{ij} be the support of D_{ij} for each player i and item j, $V_i = \times_{j\in M} V_{ij}$ and $V = \times_{i\in N} V_i$. In the optimal BIC mechanism, when player i bids v_i, let $\pi_{ij}(v_i)$ be the probability for him to get item j and $p_i(v_i)$ be his expected payment, taken over the randomness of the other players' values and the randomness of the mechanism. Let $\pi = (\pi_{ij}(v_i))_{i\in N, j\in M, v_i\in V_i}$ and $p = (p_i(v_i))_{i\in N, v_i\in V_i}$. The pair (π, p) is called the *reduced form* (of the optimal BIC mechanism) [10].

Denote by $\tilde{\varphi}_{ij}(v_{ij})$ Myerson's (ironed) virtual value when player i's value on item j is v_{ij}. For any value sub-profile v_{-i} of the players other than i, let $\beta_{ij}(v_{-i}) = \max_{i'\neq i} v_{i'j}$: that is, the highest bid on item j excluding player i. Moreover, let

$r_{ij}(v_{-i}) = \max_{x \geq \beta_{ij}(v_{-i})}\{x \cdot \Pr_{v_{ij}\sim D_{ij}}[v_{ij} \geq x]\}$, $r_i(v_{-i}) = \sum_j r_{ij}(v_{-i})$, $r_i = \mathbb{E}_{v_{-i}\sim D_{-i}}[r_i(v_{-i})]$, and finally $r = \sum_i r_i$. Note that r is the expected revenue by running the 1-look-ahead mechanism of [41] for each item separately, and $r \leq \text{Rev}(\mathcal{M}_{IM}(\mathcal{I}))$.

Next, we use a different method from [12] to partition each player i's value space V_i into $m+1$ subsets. More precisely, given $\delta > 0$ and v_{-i}, let $R_0^{(v_{-i})} = \{v_i \in V_i \mid v_{ij} < (1+\delta)\beta_{ij}(v_{-i}), \forall j\}$. For any $v_i \notin R_0^{(v_{-i})}$, let $j = \arg\max\{v_{ij} - (1+\delta)\beta_{ij}(v_{-i})\}$ with ties broken lexicographically, and add v_i to the set $R_j^{(v_{-i})}$: note that $v_{ij} - (1+\delta)\beta_{ij}(v_{-i}) \geq 0$ in this case. Similar to Theorem 3 of [12], the optimal BIC revenue can be upper-bounded by the sum of the following terms, where $D_i(v_i)$ and $D_{-i}(v_{-i})$ are respectively the probabilities of v_i and v_{-i} under D, and $\mathbf{I}$ is the indicator function:

$$\text{OPT}(\mathcal{I}) \leq \text{Single} + \text{Under} + \text{Over} + \text{Tail} + \text{Core}, \tag{16}$$

where

$$\text{Single} = \sum_i \sum_{v_i \in V_i} \sum_j D_i(v_i) \cdot \pi_{ij}(v_i) \cdot \tilde{\varphi}_{ij}(v_{ij}) \cdot \Pr_{v_{-i}\sim D_{-i}}[v_i \in R_j^{(v_{-i})}],$$

$$\text{Under} = \sum_i \sum_{v_i \in V_i} \sum_j D_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i}\in V_{-i}} v_{ij} \cdot D_{-i}(v_{-i})\mathbf{I}_{v_{ij} < (1+\delta)\beta_{ij}(v_{-i})},$$

$$\text{Over} = \sum_i \sum_{v_i \in V_i} \sum_j D_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i}\in V_{-i}} (1+\delta)\beta_{ij}(v_{-i}) D_{-i}(v_{-i})\mathbf{I}_{v_{ij} \geq (1+\delta)\beta_{ij}(v_{-i})},$$

$$\begin{aligned}\text{Tail} = \sum_i \sum_{v_{-i}\in V_{-i}} D_{-i}(v_{-i}) \sum_j \sum_{v_{ij} > (1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})} D_{ij}(v_{ij}) \cdot (v_{ij} - (1+\delta)\beta_{ij}(v_{-i})) \\ \cdot \Pr_{v_{i,-j}\sim D_{i,-j}}[\exists k \neq j, v_{ik} - (1+\delta)\beta_{ik}(v_{-i}) \geq v_{ij} - (1+\delta)\beta_{ij}(v_{-i})],\end{aligned}$$

and

$$\text{Core} = \sum_i \sum_{v_{-i}\in V_{-i}} D_{-i}(v_{-i}) \sum_j \sum_{(1+\delta)\beta_{ij}(v_{-i}) \leq v_{ij} \leq (1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})} D_{ij}(v_{ij}) \cdot (v_{ij} - (1+\delta)\beta_{ij}(v_{-i})).$$

In the following, we bound these terms in Inequality (16) separately. Note that when $\mathcal{M}_{EVIM}$ uses the value-query algorithm $\mathcal{A}_V$ to learn a distribution, the parameters are also set to be H and $\delta = \sqrt{\epsilon + 1} - 1$. Thus, applying Theorem 1 to each item, we have

$$\text{Rev}(\mathcal{M}_{IM}(\mathcal{I})) \leq (1+\delta)\text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I})).$$

For the terms Single, Under, Over and Tail, we are able to upper-bound them using $\text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I}))$. Following Lemma 13 of [12], although the term Single has changed from its original form, we still have

$$\begin{aligned}\text{Single} &= \sum_i \sum_{v_i \in V_i} \sum_j D_i(v_i) \cdot \pi_{ij}(v_i) \cdot \tilde{\varphi}_{ij}(v_{ij}) \cdot \Pr_{v_{-i}\sim D_{-i}}[v_i \in R_j^{(v_{-i})}] \\ &\leq \text{Rev}(\mathcal{M}_{IM}(\mathcal{I})) \leq (1+\delta)\text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I})).\end{aligned} \tag{17}$$

Next, using Lemmas 14 and 15 of [12], we upper-bound the term Under as follows:

$$\begin{aligned}\text{Under} &= \sum_i \sum_{v_i \in V_i} \sum_j D_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i}\in V_{-i}} D_{-i}(v_{-i}) \cdot v_{ij} \cdot \mathbf{I}_{v_{ij} < (1+\delta)\beta_{ij}(v_{-i})} \\ &= \sum_i \sum_{v_i \in V_i} \sum_j D_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i}\in V_{-i}} D_{-i}(v_{-i}) \cdot v_{ij} \cdot (\mathbf{I}_{v_{ij} < \beta_{ij}(v_{-i})} + \mathbf{I}_{\beta_{ij}(v_{-i}) \leq v_{ij} < (1+\delta)\beta_{ij}(v_{-i})}) \\ &\leq \sum_i \sum_{v_i \in V_i} \sum_j D_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i}\in V_{-i}} D_{-i}(v_{-i}) \cdot (v_{ij} \cdot \mathbf{I}_{v_{ij} < \beta_{ij}(v_{-i})} + (1+\delta)\beta_{ij}(v_{-i}) \cdot \mathbf{I}_{v_{ij} \geq \beta_{ij}(v_{-i})}) \\ &\leq \text{Rev}(\mathcal{M}_{IM}(\mathcal{I})) + (1+\delta)\text{Rev}(\mathcal{M}_{IM}(\mathcal{I})) \leq 2(1+\delta)^2\text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I})).\end{aligned}$$

The second inequality above is by Lemmas 14 and 15 of [12], which respectively upper-bound the term Over and the term Under in the original setting. Indeed, we split our term Under into the sum of the original terms Under and Over. Using the above equation, the approximation ratio to $\text{OPT}(\mathcal{I})$ will be $9(1+\epsilon)$ eventually. To get the desired $8(1+\epsilon)$-approximation, we prove a variant of Lemma 15 of [12], which directly upper-bounds our term Under as

$$\text{Under} \leq (1+\delta)\text{Rev}(\mathcal{M}_{IM}(\mathcal{I})) \leq (1+\delta)^2\text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I})). \tag{18}$$

The actual proof of this alternative lemma is tedious and does not provide new insights to our result, thus has been omitted.

Next, we upper-bound the term Over:

$$\begin{aligned}\text{Over} &= \sum_i \sum_{v_i \in V_i} \sum_j D_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i} \in V_{-i}} (1+\delta)\beta_{ij}(v_{-i}) D_{-i}(v_{-i}) \mathbf{I}_{v_{ij} \ge (1+\delta)\beta_{ij}(v_{-i})} \\ &\le (1+\delta) \sum_i \sum_{v_i \in V_i} \sum_j D_i(v_i) \cdot \pi_{ij}(v_i) \cdot \sum_{v_{-i} \in V_{-i}} \beta_{ij}(v_{-i}) D_{-i}(v_{-i}) \mathbf{I}_{v_{ij} \ge \beta_{ij}(v_{-i})} \\ &\le (1+\delta)\text{Rev}(\mathcal{M}_{IM}(\mathcal{I})) \le (1+\delta)^2 \text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I})). \end{aligned} \tag{19}$$

The second inequality above is by Lemma 14 of [12].

Next, we upper-bound the term Tail, which is similar to the analysis of [12], but with the threshold price $\beta_{ij}(v_{-i})$ scaled up by a factor of $(1+\delta)$.

$$\begin{aligned}\text{Tail} &= \sum_i \sum_{v_{-i} \in V_{-i}} D_{-i}(v_{-i}) \sum_j \sum_{v_{ij} > (1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})} D_{ij}(v_{ij}) \cdot (v_{ij} - (1+\delta)\beta_{ij}(v_{-i})) \\ &\qquad \cdot \Pr_{v_{i,-j} \sim D_{i,-j}} [\exists k \ne j, v_{ik} - (1+\delta)\beta_{ik}(v_{-i}) \ge v_{ij} - (1+\delta)\beta_{ij}(v_{-i})] \\ &\le \sum_i \sum_{v_{-i} \in V_{-i}} D_{-i}(v_{-i}) \sum_j \sum_{v_{ij} > (1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})} D_{ij}(v_{ij}) \cdot (v_{ij} - (1+\delta)\beta_{ij}(v_{-i})) \\ &\qquad \cdot \Pr_{v_{i,-j} \sim D_{i,-j}} [\exists k \ne j, v_{ik} - \beta_{ik}(v_{-i}) \ge v_{ij} - (1+\delta)\beta_{ij}(v_{-i})] \\ &\le \sum_i \sum_{v_{-i} \in V_{-i}} D_{-i}(v_{-i}) \sum_j \sum_{v_{ij} > (1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})} D_{ij}(v_{ij}) \cdot (v_{ij} - (1+\delta)\beta_{ij}(v_{-i})) \\ &\qquad \cdot \sum_{k=1}^m \Pr_{v_{ik} \sim D_{ik}} [v_{ik} \ge v_{ij} - (1+\delta)\beta_{ij}(v_{-i}) + \beta_{ik}(v_{-i})] \\ &\le \sum_i \sum_{v_{-i} \in V_{-i}} D_{-i}(v_{-i}) \sum_j \sum_{v_{ij} > (1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})} D_{ij}(v_{ij}) \\ &\qquad \cdot \sum_{k=1}^m (v_{ij} - (1+\delta)\beta_{ij}(v_{-i}) + \beta_{ik}(v_{-i})) \Pr_{v_{ik} \sim D_{ik}} [v_{ik} \ge v_{ij} - (1+\delta)\beta_{ij}(v_{-i}) + \beta_{ik}(v_{-i})] \\ &\le \sum_i \sum_{v_{-i} \in V_{-i}} D_{-i}(v_{-i}) \sum_j \sum_{v_{ij} > (1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})} D_{ij}(v_{ij}) \sum_{k=1}^m r_{ik}(v_{-i}) \\ &= \sum_i \sum_{v_{-i} \in V_{-i}} D_{-i}(v_{-i}) \sum_j r_i(v_{-i}) \sum_{v_{ij} > (1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})} D_{ij}(v_{ij}) \\ &\le \sum_i \sum_{v_{-i} \in V_{-i}} D_{-i}(v_{-i}) \sum_j ((1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})) \cdot \Pr_{v_{ij} \sim D_{ij}} [v_{ij} > (1+\delta)\beta_{ij}(v_{-i}) + r_i(v_{-i})] \\ &\le \sum_i \sum_{v_{-i} \in V_{-i}} D_{-i}(v_{-i}) \sum_j r_{ij}(v_{-i}) = \sum_i \sum_{v_{-i} \in V_{-i}} D_{-i}(v_{-i}) r_i(v_{-i}) = \sum_i r_i \\ &= r \le \text{Rev}(\mathcal{M}_{IM}(\mathcal{I})) \le (1+\delta)\text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I})). \end{aligned} \tag{20}$$

The second inequality above is by union bound. The fourth and sixth inequalities use twice the definition of $r_{ij}(v_{-i})$, which sets the optimal price to maximize the expected revenue generated by selling item j to i. The second equality is by the definition of $r_i(v_{-i})$.

Finally, we upper-bound the term Core. The Core part is the most complicated, and we use $\mathcal{M}_{EVBVCG}$ and $\mathcal{M}_{EVIM}$ together to upper-bound it. To do so, below we rewrite Core into a different form. Similar to [12], arbitrarily fixing v_{-i} and letting $v_{ij} \sim D_{ij}$, define the following two new random variables, which again scale the threshold price $\beta_{ij}(v_{-i})$ up by a factor of $(1+\delta)$:

$$b_{ij}(v_{-i}) = (v_{ij} - (1+\delta)\beta_{ij}(v_{-i})) \mathbf{I}_{v_{ij} \ge (1+\delta)\beta_{ij}(v_{-i})},$$

and

$$c_{ij}(v_{-i}) = b_{ij}(v_{-i}) \mathbf{I}_{b_{ij}(v_{-i}) \le r_i(v_{-i})}.$$

Therefore, we have

$$\text{Core} = \sum_i \sum_{v_{-i} \in V_{-i}} D_{-i}(v_{-i}) \sum_j \mathbb{E}_{v_{ij} \sim D_{ij}}[c_{ij}(v_{-i})].$$

Letting $e_i(v_{-i}) = \sum_j \mathbb{E}_{v_{ij} \sim D_{ij}}[c_{ij}(v_{-i})] - 2r_i(v_{-i})$, following the proof of Lemma 12 in [12], we still have

$$\Pr[\sum_j b_{ij}(v_{-i}) \geq e_i(v_{-i})] \geq \frac{1}{2}.$$

In the following, we use the revenue of mechanisms $\mathcal{M}_{EVBVCG}$ and $\mathcal{M}_{EVIM}$ to bound the Core. To do so, first note that by the construction of mechanism $\mathcal{M}_{EVBVCG}$,

$$\text{Rev}(\mathcal{M}_{EVBVCG}(\mathcal{I})) = \text{Rev}(\mathcal{M}_{BVCG}(\mathcal{I}; D')).$$

Let V'_{ij} be the support of D'_{ij}, $V'_i = \times_{j \in M} V'_{ij}$, $V' = \times_{i \in N} V'_i$. As before, given $v_i \sim D_i$, denote by $v'_i \in V'_i$ the value vector obtained by rounding v_i down to the support of D'_i. That is, each v'_{ij} is the largest value in V'_{ij} that is less than or equal to v_{ij}. Then,

$$\begin{aligned}\text{Rev}(\mathcal{M}_{BVCG}(\mathcal{I}; D')) &\geq \sum_i \mathbb{E}_{v_{-i} \sim D_{-i}} \mathbb{E}_{v_i \sim D_i} \text{Rev}(\mathcal{M}_{BVCG}(v'_i, v_{-i}; D')) \\ &= \sum_i \mathbb{E}_{v_{-i} \sim D_{-i}} \mathbb{E}_{v'_i \sim D'_i} \text{Rev}(\mathcal{M}_{BVCG}(v'_i, v_{-i}; D')).\end{aligned}$$

The inequality is because each player i can potentially buy item j only when j is in his winning set (i.e., he is the player with highest value for j), and i's winning set under v'_i is a subset of his wining set under v_i. Moreover, the entry fee of i is the same under both (v_i, v_{-i}) and (v'_i, v_{-i}), as it only depends on D'_i and v_{-i}. Thus the revenue inside the expectation does not increase when v_i is replaced by v'_i. The equality is again because drawing v_i from D_i and then rounding down to v'_i is equivalent to drawing v'_i from D'_i directly.

Next, we lower-bound $\sum_i \mathbb{E}_{v_{-i} \sim D_{-i}} \mathbb{E}_{v'_i \sim D'_i} \text{Rev}(\mathcal{M}_{BVCG}(v'_i, v_{-i}; D'))$. As before, arbitrarily fixing v_{-i} and letting $v'_{ij} \sim D'_{ij}$, define

$$b'_{ij}(v_{-i}) = (v'_{ij} - \beta_{ij}(v_{-i})) \mathbf{I}_{v'_{ij} \geq \beta_{ij}(v_{-i})}.$$

Note that $b'_{ij}(v_{-i})$ is a random variable that represents player i's utility in the second price mechanism on item j with value $v'_{ij} \sim D'_{ij}$, when the other players' bids are $v_{-i,j}$. Also note that $\mathcal{M}_{BVCG}$ uses the optimal entry fee for each i with respect to v_{-i} and D', which generates expected revenue from i (over D'_i) greater than or equal to that by using the following entry fee,

$$e'_i(v_{-i}) = \frac{e_i(v_{-i})}{1+\delta}.$$

Now we show player i accepts the entry fee $e'_i(v_{-i})$ with probability at least $\frac{1}{2}$. Indeed, for any v_i and the corresponding v'_i,

$$\begin{aligned}\sum_j b'_{ij}(v_{-i}) &= \sum_j (v'_{ij} - \beta_{ij}(v_{-i})) \mathbf{I}_{v'_{ij} \geq \beta_{ij}(v_{-i})} \geq \sum_j (\frac{v_{ij}}{1+\delta} - \beta_{ij}(v_{-i})) \mathbf{I}_{\frac{v_{ij}}{1+\delta} \geq \beta_{ij}(v_{-i})} \\ &= \frac{1}{1+\delta} \sum_j (v_{ij} - (1+\delta)\beta_{ij}(v_{-i})) \mathbf{I}_{v_{ij} \geq (1+\delta)\beta_{ij}(v_{-i})} = \frac{1}{1+\delta} \sum_j b_{ij}(v_{-i}).\end{aligned}$$

The inequality is because $v'_{ij} \geq \frac{v_{ij}}{1+\delta}$, and because $\frac{v_{ij}}{1+\delta} \geq \beta_{ij}(v_{-i})$ implies $v'_{ij} \geq \beta_{ij}(v_{-i})$. Therefore

$$\begin{aligned}\Pr_{v'_i \sim D'_i}[\sum_j b'_{ij}(v_{-i}) \geq e'_i(v_{-i})] &\geq \Pr_{v_i \sim D_i}[\frac{1}{1+\delta} \sum_j b_{ij}(v_{-i}) \geq \frac{e_i(v_{-i})}{1+\delta}] \\ &= \Pr_{v_i \sim D_i}[\sum_j b_{ij}(v_{-i}) \geq e_i(v_{-i})] \geq \frac{1}{2},\end{aligned}$$

as desired. Thus we have

$$\begin{aligned}
&\text{Rev}(\mathcal{M}_{EVBVCG}(\mathcal{I})) \\
&\geq \sum_i \mathbb{E}_{v_{-i}\sim D_{-i}} \mathbb{E}_{v_i' \sim D_i'} \text{Rev}(\mathcal{M}_{BVCG}(v_i', v_{-i}; D')) \geq \frac{1}{2} \sum_i \sum_{v_{-i}\in V_{-i}} D_{-i}(v_{-i}) \cdot \frac{e_i(v_{-i})}{1+\delta} \\
&= \frac{1}{2(1+\delta)} \sum_i \sum_{v_{-i}\in V_{-i}} D_{-i}(v_{-i}) \left(\sum_j \mathbb{E}_{v_{ij}\sim D_{ij}}[c_{ij}(v_{-i})] - 2r_i(v_{-i}) \right) = \frac{1}{2(1+\delta)} \text{Core} - \frac{r}{1+\delta}.
\end{aligned}$$

That is,

$$\begin{aligned}
\text{Core} &\leq 2(1+\delta)\text{Rev}(\mathcal{M}_{EVBVCG}(\mathcal{I})) + 2r \\
&\leq 2(1+\delta)\left[\text{Rev}(\mathcal{M}_{EVBVCG}(\mathcal{I})) + \text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I}))\right]. \qquad (21)
\end{aligned}$$

Combining Inequalities (16), (17), (18), (19), (20) and (21),

$$\begin{aligned}
\text{OPT}(\mathcal{I}) &\leq (1+\delta)^2 \left(2\text{Rev}(\mathcal{M}_{EVBVCG}(\mathcal{I})) + 6\text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I}))\right) \\
&= (1+\epsilon)\left(2\text{Rev}(\mathcal{M}_{EVBVCG}(\mathcal{I})) + 6\text{Rev}(\mathcal{M}_{EVIM}(\mathcal{I}))\right).
\end{aligned}$$

Accordingly, by running mechanism $\mathcal{M}_{EVBVCG}$ with probability $\frac{1}{4}$ and mechanism $\mathcal{M}_{EVIM}$ with probability $\frac{3}{4}$, the expected revenue of mechanism $\mathcal{M}_{EVA}$ is

$$\text{Rev}(\mathcal{M}_{EVA}(\mathcal{I})) \geq \frac{1}{8(1+\epsilon)} \text{OPT}(\mathcal{I}).$$

This finishes the proof of Theorem 8. □

Theorem 10 (restated). *$\forall \epsilon > 0$, any additive instance $\mathcal{I} = (N, M, D)$ satisfying Small-Tail Assumption, $\mathcal{M}_{EQA}$ is DSIC, has query complexity $O(-m^2 n \log_{1+\frac{\epsilon}{5}} h(\frac{\epsilon}{10(1+\epsilon)})$, and $\text{Rev}(\mathcal{M}_{EQA}(\mathcal{I})) \geq \frac{\text{OPT}(\mathcal{I})}{8(1+\epsilon)}$.*

Before proving Theorem 10, we first analyze mechanism $\mathcal{M}_{EQBVCG}$, and we have the following new lemma.

Lemma 6. *$\forall \epsilon > 0$, for any additive Bayesian instance $\mathcal{I} = (N, M, D)$ satisfying Small-Tail Assumption, $\mathcal{M}_{EQBVCG}$ is DSIC, has query complexity $O(-m^2 n \log_{1+\frac{\epsilon}{5}} h(\frac{\epsilon}{10(1+\epsilon)}))$, and*

$$\text{Rev}(\mathcal{M}_{EQBVCG}(\mathcal{I})) \geq \frac{1}{1+\frac{\epsilon}{5}} \left(\text{Rev}(\mathcal{M}_{BVCG}(\mathcal{I})) - \frac{\epsilon}{10(1+\epsilon)} \text{OPT}(\mathcal{I}) \right).$$

Proof. First, mechanism $\mathcal{M}_{EQBVCG}$ is DSIC because $\mathcal{M}_{BVCG}$ is DSIC. The query complexity is also immediate.

We now focus on the revenue of this mechanism. We explicitly write $\mathcal{M}_{BVCG}(\mathcal{I}; D')$ to emphasize the fact that the seller runs mechanism $\mathcal{M}_{BVCG}$ on the true valuation profile $v \sim D$, but uses D' to compute the entry fees e_i. Given a player i and a valuation profile v, $p_i(v_i, D_i, v_{-i})$ is the price for i under D_i: that is,

$$p_i(v_i, D_i, v_{-i}) = \mathbf{I}_{\sum_{j: v_{ij} \geq \beta_{ij}} (v_{ij} - \beta_{ij}) \geq e(D_i, v_{-i})} (e(D_i, v_{-i}) + \sum_j \beta_{ij} \mathbf{I}_{v_{ij} \geq \beta_{ij}}),$$

where we omit v_{-i} in $\beta_{ij}(v_{-i})$ when v_{-i} is clear from the context.[10] The price $p_i(v_i, D_i', v_{-i})$ is similarly defined. By the definition of the mechanism, we have

$$\text{Rev}(\mathcal{M}_{EQBVCG}(\mathcal{I})) = \text{Rev}(\mathcal{M}_{BVCG}(\mathcal{I}; D')) = \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i \sim D_i} p_i(v_i, D_i', v_{-i}). \qquad (22)$$

Next, let V_{ij}' be the support of D_{ij}', $V_i' = \times_{j\in M} V_{ij}'$, round v_i down to the closest valuation v_i' in V_i' and compare the two valuation profiles (v_i', v_{-i}) and (v_i, v_{-i}). By definition, $v_{ij}' \geq \beta_{ij}$ implies $v_{ij} \geq \beta_{ij}$. Moreover, the entry fee of i is the same under both valuation profiles, as it only depends on D_i' and v_{-i}. Similarly, the reserve price β_{ij} is the same for any item j. Thus we have $e(D_i', v_{-i}) + \sum_j \beta_{ij} \mathbf{I}_{v_{ij} \geq \beta_{ij}} \geq e(D_i', v_{-i}) + \sum_j \beta_{ij} \mathbf{I}_{v_{ij}' \geq \beta_{ij}}$ and $\mathbf{I}_{\sum_{j: v_{ij} \geq \beta_{ij}} (v_{ij} - \beta_{ij}) \geq e(D_i, v_{-i})} \geq \mathbf{I}_{\sum_{j: v_{ij} \geq \beta_{ij}} (v_{ij}' - \beta_{ij}) \geq e(D_i', v_{-i})}$. Therefore

$$\mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i \sim D_i} p_i(v_i, D_i', v_{-i}) \geq \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i \sim D_i} p_i(v_i', D_i', v_{-i}) = \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i' \sim D_i'} p_i(v_i', D_i', v_{-i}), \qquad (23)$$

[10] If there are ties in the players' values, then we distinguish between β_{ij}^+ and β_{ij}^-, depending on the identity of the player with the highest bid for j in $N \setminus \{i\}$.

where the equality is again because drawing v_i from D_i and then rounding down to v'_i is equivalent to drawing v'_i from D'_i directly.

In mechanism $\mathcal{M}_{BVCG}$, given v_{-i} and D'_i, $e(D'_i, v_{-i})$ is the optimal entry fee for maximizing the expected revenue generated from i, where the expectation is taken over D'_i. Accordingly,

$$\mathop{\mathbb{E}}_{v'_i \sim D'_i} p_i(v'_i, D'_i, v_{-i}) \geq \mathop{\mathbb{E}}_{v'_i \sim D'_i} p_i(v'_i, D_i, v_{-i}). \tag{24}$$

Combining Equations (22), (23) and (24), we have

$$\mathrm{Rev}(\mathcal{M}_{EQBVCG}(\mathcal{I})) \geq \sum_i \mathop{\mathbb{E}}_{v_{-i} \sim D_{-i}} \mathop{\mathbb{E}}_{v'_i \sim D'_i} p_i(v'_i, D_i, v_{-i}). \tag{25}$$

Thus we will use $\sum_i \mathop{\mathbb{E}}_{v_{-i} \sim D_{-i}} \mathop{\mathbb{E}}_{v'_i \sim D'_i} p_i(v'_i, D_i, v_{-i})$ to upper-bound $\mathrm{Rev}(\mathcal{M}_{BVCG}(\mathcal{I}))$.

To do so, first, for any player i, item j and value v_{ij}, if $v_{ij} < v'_{ij;k}$ where $v'_{ij;k}$ is the largest value in V'_{ij}, then denote by $\overline{v}_{ij}$ the smallest value in V'_{ij} that is strictly larger than v_{ij}; otherwise, let $\overline{v}_{ij} = v_{ij}$. Moreover, denote by $\underline{v}_{ij}$ the largest value in V'_{ij} that is weakly smaller than v_{ij}. The valuation $\overline{v}_i$ and $\underline{v}_i$ are defined correspondingly given v_i. Then We have

$$\begin{aligned}
&\mathrm{Rev}(\mathcal{M}_{BVCG}(\mathcal{I})) = \sum_i \mathop{\mathbb{E}}_{v_{-i} \sim D_{-i}} \mathop{\mathbb{E}}_{v_i \sim D_i} p_i(v_i, D_i, v_{-i}) \\
&= \sum_i \mathop{\mathbb{E}}_{v_{-i} \sim D_{-i}} \mathop{\mathbb{E}}_{v_i \sim D_i} \mathbf{I}_{\sum_{j:v_{ij} \geq \beta_{ij}} (v_{ij} - \beta_{ij}) \geq e(D_i, v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij} \mathbf{I}_{v_{ij} \geq \beta_{ij}} \right) \\
&= \sum_i \mathop{\mathbb{E}}_{v_{-i} \sim D_{-i}} \mathop{\mathbb{E}}_{v_i \sim D_i} \mathbf{I}_{\forall j, q_{ij}(v_{ij}) > \epsilon_1} \mathbf{I}_{\sum_{j:v_{ij} \geq \beta_{ij}} (v_{ij} - \beta_{ij}) \geq e(D_i, v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij} \mathbf{I}_{v_{ij} \geq \beta_{ij}} \right) \\
&+ \sum_i \mathop{\mathbb{E}}_{v_{-i} \sim D_{-i}} \mathop{\mathbb{E}}_{v_i \sim D_i} \mathbf{I}_{\exists j, q_{ij}(v_{ij}) \leq \epsilon_1} \mathbf{I}_{\sum_{j:v_{ij} \geq \beta_{ij}} (v_{ij} - \beta_{ij}) \geq e(D_i, v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij} \mathbf{I}_{v_{ij} \geq \beta_{ij}} \right).
\end{aligned} \tag{26}$$

Below we upper-bound the last two lines in Equation (26) separately. For the first part, we have

$$\begin{aligned}
&\sum_i \mathop{\mathbb{E}}_{v_{-i} \sim D_{-i}} \mathop{\mathbb{E}}_{v_i \sim D_i} \mathbf{I}_{\forall j, q_{ij}(v_{ij}) > \epsilon_1} \mathbf{I}_{\sum_{j:v_{ij} \geq \beta_{ij}} (v_{ij} - \beta_{ij}) \geq e(D_i, v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij} \mathbf{I}_{v_{ij} \geq \beta_{ij}} \right) \\
&\leq \sum_i \mathop{\mathbb{E}}_{v_{-i} \sim D_{-i}} \mathop{\mathbb{E}}_{v_i \sim D_i} \mathbf{I}_{\forall j, q_{ij}(v_{ij}) > \epsilon_1} \mathbf{I}_{\sum_{j:\overline{v}_{ij} \geq \beta_{ij}} (\overline{v}_{ij} - \beta_{ij}) \geq e(D_i, v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij} \mathbf{I}_{\overline{v}_{ij} \geq \beta_{ij}} \right) \\
&= \sum_i \mathop{\mathbb{E}}_{v_{-i} \sim D_{-i}} \sum_{u_i \in V'_i : \sum_{j:u_{ij} \geq \beta_{ij}} (u_{ij} - \beta_{ij}) \geq e(D_i, v_{-i})} \Pr_{v_i \sim D_i} [\overline{v}_i = u_i] \left(e(D_i, v_{-i}) + \sum_j \beta_{ij} \mathbf{I}_{u_{ij} \geq \beta_{ij}} \right).
\end{aligned} \tag{27}$$

The inequality above is because $v_{ij} \leq \overline{v}_{ij}$ for each player i and item j, which implies $\mathbf{I}_{\sum_{j:v_{ij} \geq \beta_{ij}} (v_{ij} - \beta_{ij}) \geq e(D_i, v_{-i})} \leq \mathbf{I}_{\sum_{j:\overline{v}_{ij} \geq \beta_{ij}} (\overline{v}_{ij} - \beta_{ij}) \geq e(D_i, v_{-i})}$ and $\sum_j \beta_{ij} \mathbf{I}_{v_{ij} \geq \beta_{ij}} \leq \sum_j \beta_{ij} \mathbf{I}_{\overline{v}_{ij} \geq \beta_{ij}}$.
Next, by the definition of the quantile vector q, for any $u_{ij} \in V'_{ij}$ we have

$$\Pr_{v_{ij} \sim D_{ij}} [\overline{v}_{ij} = u_{ij}] \leq (1 + \delta) \Pr_{v_{ij} \sim D_{ij}} [\underline{v}_{ij} = u_{ij}].$$

Indeed, when $u_{ij} = v'_{ij;0}$, $\Pr[v_{ij} < u_{ij}] = 0 < (1 + \delta)(1 - \epsilon_1(1 + \delta)^{k-1}) = (1 + \delta)(q_0 - q_1) = (1 + \delta) \Pr[v_{ij} \in [v'_{ij;0}, v'_{ij;1})]$. When $u_{ij} = v'_{ij;l}$ with $0 < l < k$, $\Pr(v_{ij} \in [v'_{ij;l-1}, v'_{ij;l})) = q_{l-1} - q_l \leq (1 + \delta)q_l - q_l = \delta q_l = (1 + \delta)\delta q_{l+1} = (1 + \delta)((1 + \delta)q_{l+1} - q_{l+1}) = (1 + \delta)(q_l - q_{l+1}) = (1 + \delta) \Pr[v_{ij} \in [v'_{ij;l}, v'_{ij;l+1})]$. And when $u_{ij} = v'_{ij;k}$, $\Pr[v_{ij} \in [v'_{ij;k-1}, v'_{ij;k}]] = q_{k-1} - q_k = \delta\epsilon_1 < \epsilon_1 = \Pr[v_{ij} \geq v'_{ij;k}]$. Since all distributions are independent, for any $u_i \in V'_i$ we have

$$\Pr_{v_i \sim D_i} [\overline{v}_i = u_i] \leq (1 + \delta)^m \Pr_{v_i \sim D_i} [\underline{v}_i = u_i]. \tag{28}$$

Combining Equations (27) and (28), we have

$$\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i\sim D_i} \mathbf{I}_{\forall j, q_{ij}(v_{ij})>\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(D_i,v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}}\right)$$
$$\leq \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \sum_{u_i\in V_i': \sum_{j:u_{ij}\geq\beta_{ij}}(u_{ij}-\beta_{ij})\geq e(D_i,v_{-i})} (1+\delta)^m \cdot \Pr_{v_i\sim D_i}[\underline{v}_i = u_i] \cdot \left(e(D_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{u_{ij}\geq\beta_{ij}}\right)$$
$$= (1+\delta)^m \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i'\sim D_i'} \mathbf{I}_{\sum_{j:v'_{ij}\geq\beta_{ij}}(v'_{ij}-\beta_{ij})\geq e(D_i,v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v'_{ij}\geq\beta_{ij}}\right)$$
$$= (1+\delta)^m \sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i'\sim D_i'} p_i(v_i', D_i, v_{-i}) \leq (1+\delta)^m \mathrm{Rev}(\mathcal{M}_{ECBVCG}(\hat{\mathcal{I}})). \tag{29}$$

The first equality above holds because drawing v_i from D_i and rounding down to the support of D_i' is equivalent to drawing v_i' from D_i'. The second equality is by the definition of $p_i(v_i', D_i, v_{-i})$, and the last inequality holds by Equation (25).

By Equations (26) and (29), we have

$$\mathrm{Rev}(\mathcal{M}_{BVCG}(\mathcal{I}))$$
$$\leq (1+\delta)^m \mathrm{Rev}(\mathcal{M}_{EQBVCG}(\mathcal{I}))$$
$$+\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i\sim D_i} \mathbf{I}_{\exists j, q_{ij}(v_{ij})\leq\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(D_i,v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}}\right). \tag{30}$$

For the last line of Equation (30), we have

$$\sum_i \mathop{\mathbb{E}}_{v_{-i}\sim D_{-i}} \mathop{\mathbb{E}}_{v_i\sim D_i} \mathbf{I}_{\exists j, q_{ij}(v_{ij})\leq\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(D_i,v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}}\right)$$
$$= \mathop{\mathbb{E}}_{v\sim D} \sum_i \mathbf{I}_{\exists j, q_{ij}(v_{ij})\leq\epsilon_1} \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(D_i,v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}}\right)$$
$$\leq \mathop{\mathbb{E}}_{v\sim D} \mathbf{I}_{\exists i,j, q_{ij}(v_{ij})\leq\epsilon_1} \sum_i \mathbf{I}_{\sum_{j:v_{ij}\geq\beta_{ij}}(v_{ij}-\beta_{ij})\geq e(D_i,v_{-i})} \left(e(D_i, v_{-i}) + \sum_j \beta_{ij}\mathbf{I}_{v_{ij}\geq\beta_{ij}}\right)$$
$$= \mathop{\mathbb{E}}_{v\sim D} \mathbf{I}_{\exists i,j, q_{ij}(v_{ij})\leq\epsilon_1} \mathrm{Rev}(\mathcal{M}_{BVCG}(v;\mathcal{I})) \leq \frac{\epsilon}{10(1+\epsilon)}\mathrm{OPT}(\mathcal{I}). \tag{31}$$

The first inequality above is because, for each player i and valuation profile v, $\mathbf{I}_{\exists j, q_{ij}(v_{ij})\leq\epsilon_1} \leq \mathbf{I}_{\exists i,j, q_{ij}(v_{ij})\leq\epsilon_1}$. The second inequality is by the Small-Tail Assumption.

Combining Equations (30) and (31), we have

$$\mathrm{Rev}(\mathcal{M}_{BVCG}(\mathcal{I})) \leq (1+\delta)^m \mathrm{Rev}(\mathcal{M}_{EQBVCG}(\mathcal{I})) + \frac{\epsilon}{10(1+\epsilon)}\mathrm{OPT}(\mathcal{I}).$$

By the construction of Mechanism 10, $(1+\delta)^m = 1+\frac{\epsilon}{5}$. Therefore Lemma 6 holds. □

Proof of Theorem 10. First, as both $\mathcal{M}_{EQBVCG}$ and $\mathcal{M}_{EQIM}$ are DSIC, $\mathcal{M}_{EQA}$ is DSIC. Second, note that $\mathcal{M}_{EQA}$ runs both mechanisms with $\delta = (1+\frac{\epsilon}{5})^{1/m} - 1$ and $\epsilon_1 = h(\frac{\epsilon}{10(1+\epsilon)})$. To ease the analysis, when running mechanism $\mathcal{M}_{EQIM}$, let $\delta = \frac{\epsilon}{15}$ and $\epsilon_1 = h(\frac{2\epsilon}{3(5+\epsilon)})$: that is, set $\epsilon' = \frac{\epsilon}{5}$ and run mechanism $\mathcal{M}_{EQM}$ with parameter ϵ' for each item. By Theorem 2, with $O(-mn\log_{1+\frac{\epsilon}{15}} h(\frac{2\epsilon}{3(5+\epsilon)}))$ queries,

$$\mathrm{Rev}(\mathcal{M}_{EQIM}(\mathcal{I})) \geq \frac{1}{1+\frac{\epsilon}{5}}\mathrm{Rev}(\mathcal{M}_{IM}(\mathcal{I})).$$

By Lemma 6, with $O(-m^2 n\log_{1+\frac{\epsilon}{5}} h(\frac{\epsilon}{10(1+\epsilon)}))$ queries,

$$\mathrm{Rev}(\mathcal{M}_{EQBVCG}(\mathcal{I})) \geq \frac{1}{1+\frac{\epsilon}{5}}\left(\mathrm{Rev}(\mathcal{M}_{BVCG}(\mathcal{I})) - \frac{\epsilon}{10(1+\epsilon)}\mathrm{OPT}(\mathcal{I})\right).$$

Note that the total query complexity is still $O(-m^2 n\log_{1+\frac{\epsilon}{5}} h(\frac{\epsilon}{10(1+\epsilon)}))$.

Let mechanism $\mathcal{M}_{EQA}$ run $\mathcal{M}_{EQBVCG}$ with probability $\frac{1}{4}$ and $\mathcal{M}_{EQIM}$ with probability $\frac{3}{4}$. We have

$$\begin{aligned}
&\mathrm{Rev}(\mathcal{M}_{EQA}(\mathcal{I})) = \frac{1}{4}\mathrm{Rev}(\mathcal{M}_{EQBVCG}(\mathcal{I})) + \frac{3}{4}\mathrm{Rev}(\mathcal{M}_{EQIM}(\mathcal{I}))\\
&\geq \frac{1}{4(1+\frac{\epsilon}{5})}\left(\mathrm{Rev}(\mathcal{M}_{BVCG}(\mathcal{I})) - \frac{\epsilon}{10(1+\epsilon)}\mathrm{OPT}(\mathcal{I})\right) + \frac{3}{4(1+\frac{\epsilon}{5})}\mathrm{Rev}(\mathcal{M}_{IM}(\mathcal{I}))\\
&\geq \frac{1}{1+\frac{\epsilon}{5}}\left(\frac{1}{4}\mathrm{Rev}(\mathcal{M}_{BVCG}(\mathcal{I})) + \frac{3}{4}\mathrm{Rev}(\mathcal{M}_{IM}(\mathcal{I})) - \frac{\epsilon}{10(1+\epsilon)}\mathrm{OPT}(\mathcal{I})\right)\\
&\geq \frac{1}{1+\frac{\epsilon}{5}}\left(\frac{1}{8}\mathrm{OPT}(\mathcal{I}) - \frac{\epsilon}{10(1+\epsilon)}\mathrm{OPT}(\mathcal{I})\right) = \frac{1}{8(1+\epsilon)}\mathrm{OPT}(\mathcal{I}).
\end{aligned}$$

The last inequality above holds because $2\mathcal{M}_{BVCG}(\mathcal{I}) + 6\mathcal{M}_{IM}(\mathcal{I}) \geq \mathrm{OPT}(\mathcal{I})$ proved in [12]. Thus Theorem 10 holds. □

Appendix C. Missing proofs for Section 7

Theorem 11 (restated). *$\forall\epsilon > 0$ and $\gamma \in (0, 1)$, for any Bayesian instance $\mathcal{I} = (N, M, D)$,*

- *for single-item auctions satisfying the Small-Tail Assumption, with $\tilde{O}(h^{-2}(\frac{2\epsilon}{3(1+\epsilon)})\cdot(\frac{\epsilon}{1+\epsilon})^{-2})$ samples, mechanism $\mathcal{M}_{SM}$ achieves revenue at least $\frac{1}{1+\epsilon}\mathrm{OPT}(\mathcal{I})$ with probability at least $1-\gamma$;*
- *for unit-demand auctions satisfying the Small-Tail Assumption, with $\tilde{O}(h^{-2}(\frac{2\epsilon}{3(1+\epsilon)})\cdot(\frac{\epsilon}{1+\epsilon})^{-2})$ samples, mechanism $\mathcal{M}_{SM}$ achieves revenue at least $\frac{1}{24(1+\epsilon)}\mathrm{OPT}$ with probability at least $1-\gamma$;*
- *for additive auctions satisfying the Small-Tail Assumption, with $\tilde{O}(h^{-2}(\frac{\epsilon}{10(1+\epsilon)})(\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}})^{-2})$ samples, mechanism $\mathcal{M}_{SM}$ achieves revenue at least $\frac{1}{8(1+\epsilon)}\mathrm{OPT}$ with probability at least $1-\gamma$.*

Proof of Theorem 11. After constructing the distributions, we simply run the existing DSIC mechanisms as a Blackbox, and if the constructed distribution satisfies the property that for any quantile q_l,

$$q_{ij}(v_{ij}^{t\cdot q_{l+1}}) \geq \frac{1}{1+\frac{\epsilon}{3}}\left(q_{ij}(v_{ij}^{t\cdot q_l})\right). \tag{32}$$

all our query complexity results for single-item and unit-demand auctions directly apply here.

Since here for sampling mechanism, we slice the quantile interval uniformly, in the ideal case, the selected sampled values correspond to the desired quantiles and $D_{ij}(v_{ij}^{t\cdot q_l}) = D_{ij}(v_{ij}^{t\cdot q_{l+1}})$. However, since these samples are random, we may not obtain the ideal case. In fact, given parameter $d = \frac{12+3\epsilon}{\epsilon}$, if for any quantile q_l,

$$q_l - \frac{q_l}{d} \leq q_{ij}(v_{ij}^{t\cdot q_l}) \leq q_l + \frac{q_l}{d}, \tag{33}$$

then

$$\frac{q_{ij}(v_{ij}^{t\cdot q_{l+1}})}{q_{ij}(v_{ij}^{t\cdot q_l})} \geq \frac{q_{l+1}(1-\frac{1}{d})}{q_l(1+\frac{1}{d})} \geq \frac{\frac{1}{1+\frac{\epsilon}{6}}(1-\frac{1}{d})}{1+\frac{1}{d}} = \frac{1}{1+\frac{\epsilon}{3}},$$

for any $\epsilon > 0$, that is, Equation (32) holds. In the following, we show how many samples are enough to obtain Inequality (33).

First, we bound the probability that $v_{ij}^{t\cdot q_l}$ locates in the quantile interval $[q_l - \frac{q_l}{d}, q_l + \frac{q_l}{d}]$. Let $E_{ij,l}^{left}$ be the event that $v_{ij}^{t\cdot q_l}$ locates in the quantile interval $[0, q_l - \frac{q_l}{d}]$, and $E_{ij,l}^{right}$ be the event that $v_{ij}^{t\cdot q_l}$ locates in the quantile interval $[q_l + \frac{q_l}{d}, 1]$. Then

$$\Pr[E_{ij,l}^{left}] = \sum_{s=0}^{t-t\cdot q_l}\binom{t}{s}\left(q_l - \frac{q_l}{d}\right)^s\left(1-q_l+\frac{q_l}{d}\right)^{t-s},$$

and

$$\Pr[E_{ij,l}^{right}] = \sum_{s=0}^{t\cdot q_l}\binom{t}{s}\left(1-q_l-\frac{q_l}{d}\right)^s\left(q_l+\frac{q_l}{d}\right)^{t-s}.$$

By Chernoff's inequality and $\forall i, j, l$, letting $\Pr[E_{ij,l}^{left}]$ and $\Pr[E_{ij,l}^{right}]$ be no more than $\frac{\gamma}{2mn(k+1)}$, $t = \tilde{O}((\frac{\epsilon_1}{d})^{-2}) = \tilde{O}((\frac{\epsilon\cdot\epsilon_1}{(1+\epsilon)})^{-2})$. That is with $\tilde{O}(h^{-2}(\frac{2\epsilon}{3(1+\epsilon)})\cdot(\frac{\epsilon}{1+\epsilon})^{-2})$ samples, the probability that $v_{ij}^{t\cdot q_l}$ does not locate in the quantile interval $[q_l - \frac{q_l}{d}, q_l +$

$\frac{q_l}{d}]$ is less than $\frac{\gamma}{mn(k+1)}$. By union bound, there exists one $v_{ij}^{t \cdot q_l}$ for all $i \in [n], j \in [m], l \in [k+1]$ does not locate in the quantile interval $[q_l - \frac{q_l}{d}, q_l + \frac{q_l}{d}]$ is less than γ. Then with probability $1-\gamma$, Inequality (33) holds.

For additive auctions, if the constructed distribution satisfies the property that for any quantile q_l,

$$q_{ij}(v_{ij}^{t \cdot q_l}) - q_{ij}(v_{ij}^{t \cdot q_{l+1}}) \geq \frac{1}{(1+\frac{\epsilon}{5})^{1/m}} \left(q_{ij}(v_{ij}^{t \cdot q_{l+1}}) - q_{ij}(v_{ij}^{t \cdot q_{l+2}}) \right), \tag{34}$$

all our query complexity results for additive auctions directly apply here. In fact, if for any quantile q_l,

$$q_l - \epsilon_1 (\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}}) \leq q_{ij}(v_{ij}^{t \cdot q_l}) \leq q_l + \epsilon_1 (\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}}),$$

then,

$$\frac{q_{ij}(v_{ij}^{t \cdot q_l}) - q_{ij}(v_{ij}^{t \cdot q_{l+1}})}{q_{ij}(v_{ij}^{t \cdot q_{l+q}}) - q_{ij}(v_{ij}^{t \cdot q_{l+2}})} \geq \frac{\epsilon_1 - \epsilon_1 (1 - \frac{2}{1+(1+\frac{\epsilon}{5})^{1/m}})}{\epsilon_1 + \epsilon_1 (1 - \frac{2}{1+(1+\frac{\epsilon}{5})^{1/m}})} = \frac{1}{(1+\frac{\epsilon}{5})^{1/m}},$$

Using the same technique of applying the Chernoff's inequality, with $\tilde{O}(h^{-2}(\frac{\epsilon}{10(1+\epsilon)})(\frac{1}{2} - \frac{1}{1+(1+\frac{\epsilon}{5})^{1/m}}))^{-2})$ samples, Equation (34) holds with probability $1-\gamma$. Thus Theorem 11 holds. □

References

[1] A. Allouah, O. Besbes, Sample-based optimal pricing, 2019, Available at SSRN 3334650.
[2] P. Azar, C. Daskalakis, S. Micali, S.M. Weinberg, Optimal and efficient parametric auctions, in: 24th Symposium on Discrete Algorithms, SODA'13, 2013, pp. 596–604.
[3] P. Azar, S. Micali, Parametric digital auctions, in: 4th Innovations in Theoretical Computer Science Conference, ITCS'13, 2013, pp. 231–232.
[4] P.D. Azar, R. Kleinberg, S.M. Weinberg, Prophet inequalities with limited information, in: 25th Annual ACM-SIAM Symposium on Discrete Algorithms, SODA'14, 2014, pp. 1358–1377.
[5] M. Babaioff, Y.A. Gonczarowski, Y. Mansour, S. Moran, Are two (samples) really better than one?, in: Proceedings of the 2018 ACM Conference on Economics and Computation, 2018, p. 175.
[6] M. Babaioff, Y.A. Gonczarowski, N. Nisan, The menu-size complexity of revenue approximation, in: Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, 2017, pp. 869–877.
[7] D. Bergemann, K. Schlag, Robust monopoly pricing, J. Econ. Theory 146 (6) (2011) 2527–2543.
[8] J. Brustle, Y. Cai, C. Daskalakis, Multi-item mechanisms without item-independence: learnability via robustness, in: Proceedings of the 21st ACM Conference on Economics and Computation, 2020, pp. 715–761.
[9] Y. Cai, C. Daskalakis, Learning multi-item auctions with (or without) samples, in: 58th Symposium on Foundations of Computer Science, FOCS'17, 2017, pp. 516–527.
[10] Y. Cai, C. Daskalakis, S.M. Weinberg, An algorithmic characterization of multi-dimensional mechanisms, in: 44th Annual ACM Symposium on Theory of Computing, STOC'12, 2012, pp. 459–478.
[11] Y. Cai, C. Daskalakis, S.M. Weinberg, Optimal multi-dimensional mechanism design: reducing revenue to welfare maximization, in: 53rd Symposium on Foundations of Computer Science, FOCS'12, 2012, pp. 130–139.
[12] Y. Cai, N.R. Devanur, S.M. Weinberg, A duality based unified approach to Bayesian mechanism design, in: 48th Annual ACM Symposium on Theory of Computing, STOC'16, 2016, pp. 926–939.
[13] Y. Cai, M. Zhao, Simple mechanisms for subadditive buyers via duality, in: 49th Symposium on Theory of Computing, STOC'17, 2017, pp. 170–183.
[14] V. Carrasco, V.F. Luz, N. Kos, M. Messner, P. Monteiro, H. Moreira, Optimal selling mechanisms under moment conditions, J. Econ. Theory 177 (2018) 245–279.
[15] S. Chawla, J.D. Hartline, D.L. Malec, B. Sivan, Multi-parameter mechanism design and sequential posted pricing, in: 43rd ACM Symposium on Theory of Computing, STOC'10, 2010, pp. 311–320.
[16] S. Chawla, J.B. Miller, Mechanism design for subadditive agents via an ex ante relaxation, in: 17th Conference on Economics and Computation, EC'16, ACM, 2016, pp. 579–596.
[17] E. Che, Robust reserve pricing in auctions under mean constraints, 2019, Available at SSRN 3488222.
[18] R. Cole, T. Roughgarden, The sample complexity of revenue maximization, in: 46th Annual ACM Symposium on Theory of Computing, STOC'14, 2014, pp. 243–252.
[19] J. Cremer, R.P. McLean, Full extraction of the surplus in Bayesian and dominant strategy auctions, Econometrica 56 (6) (1988) 1247–1257.
[20] C. Daskalakis, A. Deckelbaum, C. Tzamos, Mechanism design via optimal transport, in: 14th Conference on Electronic Commerce, EC'13, 2013, pp. 269–286.
[21] C. Daskalakis, M. Zampetakis, More revenue from two samples via factor revealing sdps, in: Proceedings of the 21st ACM Conference on Economics and Computation, 2020, pp. 257–272.
[22] N.R. Devanur, Z. Huang, C.-A. Psomas, The sample complexity of auctions with side information, in: 48th Annual ACM Symposium on Theory of Computing, STOC'16, 2016, pp. 426–439.
[23] P. Dhangwatnotai, T. Roughgarden, Q. Yan, Revenue maximization with a single sample, Games Econ. Behav. 91 (2015) 318–333.
[24] S. Dobzinski, Computational efficiency requires simple taxation, in: 57th Symposium on Foundations of Computer Science, FOCS'16, 2016, pp. 209–218.
[25] C. Dwork, Differential privacy: a survey of results, in: International Conference on Theory and Applications of Models of Computation, Springer, 2008, pp. 1–19.
[26] K. Goldner, A.R. Karlin, A prior-independent revenue-maximizing auction for multiple additive bidders, in: 12th International Conference on Web and Internet Economics, WINE'16, 2016, pp. 160–173.
[27] Y.A. Gonczarowski, N. Nisan, Efficient empirical revenue maximization in single-parameter auction environments, in: 49th Symposium on Theory of Computing, STOC'17, 2017, pp. 856–868.
[28] R. Guesnerie, On taxation and incentives: further remarks on the limits to redistribution, University of Bonn, 1981, 89.

[29] C. Guo, Z. Huang, X. Zhang, Settling the sample complexity of single-parameter revenue maximization, in: Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, 2019, pp. 662–673.
[30] P.J. Hammond, Straightforward individual incentive compatibility in large economies, Rev. Econ. Stud. 46 (2) (1979) 263–282.
[31] S. Hart, N. Nisan, The menu-size complexity of auctions, in: 14th ACM Conference on Electronic Commerce, EC'13, 2013, pp. 565–566.
[32] S. Hart, N. Nisan, Approximate revenue maximization with multiple items, J. Econ. Theory 172 (2017) 313–347.
[33] S. Hart, P.J. Reny, Maximal revenue with multiple goods: nonmonotonicity and other observations, Theor. Econ. 10 (2015) 893–922.
[34] J. Hartline, S. Taggart, Sample complexity for non-truthful mechanisms, in: Proceedings of the 2019 ACM Conference on Economics and Computation, 2019, pp. 399–416.
[35] Z. Huang, Y. Mansour, T. Roughgarden, Making the most of your samples, in: 16th ACM Conference on Economics and Computation, EC'15, 2015, pp. 45–60.
[36] R. Kleinberg, S.M. Weinberg, Matroid prophet inequalities, in: 44th Annual ACM Symposium on Theory of Computing, STOC'12, 2012, pp. 123–136.
[37] Y. Li, P. Lu, H. Ye, Revenue maximization with imprecise distribution, in: Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, 2019, pp. 1582–1590.
[38] J. Morgenstern, T. Roughgarden, Learning simple auctions, in: 29th Conference on Learning Theory, COLT'16, 2016, pp. 1298–1318.
[39] R.B. Myerson, Optimal auction design, Math. Oper. Res. 6 (1) (1981) 58–73.
[40] J.-C. Rochet, The taxation principle and multi-time Hamilton-Jacobi equations, J. Math. Econ. 14 (2) (1985) 113–128.
[41] A. Ronen, On approximating optimal auctions, in: 3rd ACM Conference on Electronic Commerce, EC'01, 2001, pp. 11–17.
[42] T. Roughgarden, O. Schrijvers, Ironing in the dark, in: 17th ACM Conference on Economics and Computation, EC'16, 2016, pp. 1–18.
[43] A. Rubinstein, S.M. Weinberg, Simple mechanisms for a subadditive buyer and applications to revenue monotonicity, in: 16th ACM Conference on Economics and Computation, EC'15, 2015, pp. 377–394.
[44] A.C.-C. Yao, An n-to-1 bidder reduction for multi-item auctions and its applications, in: 26th Annual ACM-SIAM Symposium on Discrete Algorithms, SODA'15, 2015, pp. 92–109.
[45] A.C.-C. Yao, Dominant-strategy versus bayesian multi-item auctions: maximum revenue determination and comparison, in: 18th Conference on Economics and Computation, EC'17, 2017, pp. 3–20.