CSE540, Fall 2008

Homework 2

Basic Questions

B9. For any set $A \subseteq \{0,1\}^*$, A^* denotes the set of all words that can be formed as the concatenation of a finite number of words in A; in particular, the empty string λ is in A^*, and $A \subseteq A^*$. Show that if A is r.e., then A^* is also r.e.

B10. Let $A, B \subseteq \{0,1\}^*$ be two sets with the properties $A \cup B = \{0,1\}^*$ and $A \cap B \neq \emptyset$. Show that if A, B are r.e., then $A \leq_m A \cap B$.

B11. Determine whether the following statement is true or false. Give brief justification for your answer.

> *The set $\{n \mid W_n \subseteq W_{n+1}\}$ is an index set.*

B12. Determine whether the following statement is true or false. Give brief justification for your answer.

> *If A is an r.e. set, and f a total recursive function, then $f^{-1}[A] =_{defn} \{x \mid f(x)$ halts and the output value is in $A\}$ is also r.e.*

B13. Determine whether the following problem is decidable (or, recursive). Give brief justification for your answer.

> *Given the code of a (one-tape) TM M and an input string x, determine whether, during the computation of M on x, the tape head of M will move $2^{|x|}$ cells away from the starting position.*

B14. Determine whether the following problem is decidable (or, recursive). Give brief justification for your answer.

> *Given the codes of two (one-tape) TMs M and N, determine whether for some input x, they both halt and output the same value.*

Medium Questions

M6. Let $A = \{n \mid |\{0,1\}^* - W_n| \leq 2\}$, where $|B|$ denotes the size of a (finite) set. Apply the diagonalization technique to prove that A is not an r.e.

set. [Hint: For every partial recursive function, diagonalize against it at three different inputs.]

M7. Use the reduction technique to prove that set A defined in problem M6 is not an r.e. set.

For each set in problems M8–M12, determine whether it is recursive, and if the answer is no, whether it is r.e. and whether it is co-r.e. [Note: If you would like to apply Rice's theorem, make sure to argue that the set is an index set. Otherwise, show your complete proof (the diagonalization proof, or the reduction).]

M8. $\{n \mid |W_n| \geq 5\}$.

M9. $\{n \mid Q \subseteq W_n\}$, where Q denotes the set of all prime numbers.

M10. $\{n \mid W_n \subseteq Q\}$, where Q denotes the set of all prime numbers.

M11. $\{n \mid \{0,1\}^* - W_n$ is a finite set$\}$.

M12. $\{\langle x, y\rangle \mid \phi_x = \phi_y\}$.

M13. Exercise 1.31 of the textbook.

Hard Problems

H1. Two sets A, B are called *recursively separable* if there is a recursive set C such that $A \subseteq C$ and $B \subseteq \overline{C}$. Two sets are *recursively inseparable* if they are not recursively separable. Let K_n denote the set $\{x \mid \phi_x(x)$ is defined and is equal to $n\}$. Prove that, for any $n \neq m$, K_n and K_m are recursively inseparable.

H2. In this question, we consider the tiling problem. In this problem, a *colored tile* is a square tile of size 1×1 whose four sides are colored by colors selected from a finite set C. The four sides of a colored tile is clearly marked to be the up, down, left and right sides. Two colored tiles can be put in the plane next to each other if the two neighboring sides have the same color.

> TILING: Given a finite number of types $t_0, t_1, \ldots, t_n$ of colored tiles, determine whether it is possible to cover the top half of the plane (i.e, all squares $[a, a+1] \times [b, b+1]$, $a \in \mathbf{Z}$ and $b \in \mathbf{N}$) by colored tiles of these types (with infinite supply of the tiles of each type), starting with a tile of type t_0 at the square $[0,1] \times [0,1]$.

Show that the problem TILING is undecidable. [*Hint*: Encode the Turing machine configurations by colors so that the colors of each row in the top half of the plane represents one configuration, and the change of colors from one row to the next follows the instructions of a Turing machine.]