# CSE540, Fall 2009

# Homework 3

### Basic Questions

**B15**. Let $A = \{\langle n, k\rangle : n \geq 1,\ n = p \cdot q$ for some integers $p, q$ with $2 \leq p \leq k, 2 \leq q \leq n\}$. [Note: In the above definition, we assume all integers are written in the binary notation.]

(a) Show that $A$ is in $NP$.

(b) Show that if $A \in P$, then there is a polynomial-time algorithm that can find, for any given integer $n \geq 2$, its prime factorization.

**B16**. Exercise 2.1 of the textbook, with the following correction: Prove

(a) For every polynomially honest function $f \in FP$, $range(f)$ is in $NP$, and

(b) If $A \in NP$ and $A \neq \emptyset$, then $A = range(f)$ for some $f \in FP$.

**B17**. Exercise 2.2 of the textbook.

**B18**. Prove that if $A \in P$ then $A^* \in P$.
[Note: $A^*$ is defined in question B9.]

**B19**. Find a polynomial-time many-one ($\leq_m^P$) reduction from Hamiltonian Circuit to Subgraph Isomorphism.
[Note: Subgraph Isomorphism is defined in Exercise 2.11(d) of the textbook.]

**B20**. Consider the following problem:

> Large Circuit (LC): Given a graph $G = (V, E)$, determine whether there exists a simple cycle that contains at least $|V|/2$ vertices.

Find a polynomial-time many-one reduction from HC to LC.

**Medium Questions**

**M14**. Exercise 2.22, part (a), of the textbook.

**M15**. Give a direct reduction for Vertex Cover $\leq_m^P$ Subset Sum. Prove the correctness of the reduction.

For each of the problems M16 and M17, if you present a reduction $A \leq_m^P B$ to prove that $B$ is $NP$-complete, please use a problem $A$ that has been proven to be $NP$-complete in our textbook or the *Problem Solving* book. If you use any other set $A$, including those in the exercises of the two books, you need first prove that $A$ itself is $NP$-complete.

**M16**. Let $K$ be a positive integer. We say an $m \times n$ integer matrix $M$ is *row-bounded by* $K$ (or *column-bounded by* $K$) if the sum of each row of $M$ is $\leq K$ (or, respectively, the sum of each column of $M$ is $\leq K$). An $m \times n$ integer matrix $M$ is *row-column-bounded by* $K$ if, for each pair $(i, j)$, with $1 \leq i \leq m$ and $1 \leq j \leq n$, the sum of all entries in the $i$th row or in the $j$th column of $M$ is $\leq K$ (i.e., $\sum_{k=1}^{n} A[i,k] + \sum_{\ell=1}^{m} A[\ell,j] - A[i,j] \leq K$).
Prove that the following problem is $NP$-complete:

> Matrix Partition. Given a pair $(M, K)$, where $K$ is a positive integer and $M$ an $m \times n$ matrix of nonnegative integers, which is both row-bounded by $K$ and column-bounded by $K$, determine whether there exist two $m \times n$ matrices $A$ and $B$ such that
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> (1) For each pair $(i, j)$, with $1 \leq i \leq m$ and $1 \leq j \leq n$, either $(A[i,j] = M[i,j]$ and $B[i,j] = 0)$ or $(A[i,j] = 0$ and $B[i,j] = M[i,j])$;
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> (2) Both $A$ and $B$ are row-column-bounded by $K$.

**M17**. Exercise 2.20, part (c), of the textbook.

**M18**. Exercise 2.15, part (a), of the textbook.

**Hard Problems**

**H3**. Exercise 2.11, part (b), of the textbook.

**H4**. Let $F$ be a Boolean formula in 3-CNF that has clauses $C_1, C_2, \ldots, C_m$, and variables $x_1, x_2, \ldots, x_n$. We define a bipartite graph $G_F$ associated with $F$ as follows: The two vertex sets of $G_F$ are $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$, and the edge set $E$ contains all edges $\{u_i, v_j\}$ if variable $x_i$ occurs in clause $C_j$ (either as $x_i$ or as $\overline{x_i}$). Recall that a graph is planar if we can draw it on the two-dimensional plane with edges not crossing each other. Prove that the following subproblem of 3-SAT is $NP$-complete.

> Given a 3-CNF Boolean formula $F$ whose associated bipartite graph $G_F$ is planar, determine whether $F$ is satisfiable.

**H5**. Exercise 1.28, part (a), of the textbook.
[Note: Discuss, without formal proofs, the enumeration of NTMs and the existence of the nondeterministic universal TM.]