Predicate Logic

The analysis of compound statements covers key aspects of human reasoning but does not capture many important, and common, instances of reasoning that are also logically valid.

For example, consider the following argument:

> All men are mortal.
> Socrates is a man.
> ∴ Socrates is mortal.

This argument is intuitively correct, but its validity can not be captured by propositional methods.

We will next consider a formal system, called *predicate logic*, that allows for an analysis of predicates and quantified statements (such as the first hypothesis above).

Predicate Logic: Syntax

The syntax of predicate logic is specified by the following rules:

$$
\begin{array}{lll}
\langle\text{formula}\rangle & ::= & \langle\text{predsymbol}\rangle \\
 & & \mid \langle\text{predsymbol}\rangle(\langle\text{termlist}\rangle) \\
 & & \mid (\sim\langle\text{formula}\rangle) \\
 & & \mid (\langle\text{formula}\rangle \vee \langle\text{formula}\rangle) \\
 & & \mid (\forall\langle\text{declaration}\rangle, \langle\text{formula}\rangle) \\
 & & \mid (\exists\langle\text{declaration}\rangle, \langle\text{formula}\rangle) \\
\langle\text{declaration}\rangle & ::= & \langle\text{variable}\rangle \in \langle\text{domain}\rangle \\
\langle\text{term}\rangle & ::= & \langle\text{variable}\rangle \mid \langle\text{constant}\rangle \\
\langle\text{termlist}\rangle & ::= & \langle\text{term}\rangle \mid \langle\text{term}\rangle, \langle\text{termlist}\rangle \\
\langle\text{predsymbol}\rangle & ::= & P \mid Q \mid R \mid \ \ldots \\
\langle\text{constant}\rangle & ::= & a \mid b \mid c \mid \ \ldots \\
\langle\text{variable}\rangle & ::= & x \mid y \mid z \mid \ \ldots \\
\langle\text{domain}\rangle & ::= & D \mid E \mid \ \ldots
\end{array}
$$

Notation

The symbols $\forall$ and $\exists$ are called *quantifiers*.

We adopt various conventions so as to simplify expressions and increase readability:

- We delete unnecessary parentheses.
- We do not always stick to strict prefix format, but also use infix or other notation.
- We use special symbols to denote constants, predicate symbols, or domains, depending on the context.
- We also employ function symbols in the construction of terms.
- We use connectives other than $\sim$ and $\vee$.

For example, in the formula

$$\forall x \in \mathbf{Z}, \forall y \in \mathbf{Z}, [x < 0 \wedge 0 < y \rightarrow x * y < 0]$$

the letter $\mathbf{Z}$ refers to the integers, the symbol $*$ denotes multiplication, and $<$ denotes the less-than relation. (The formula expresses that the product of a negative and a positive integer is negative.)

Domains

Domains are non-empty sets and may be finite or infinite. Typical domains include sets of numbers, sets of strings, sets of lists, etc.

The letters $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ denote, respectively, the set of natural numbers, the set of integers, the set of rational numbers, and the set of real numbers.

In order to evaluate a predicate logic formula one has to specify which domains are intended for the different variables and indicate the meaning of constants and predicate symbols.

A formula may contain both quantified variables and non-quantified variables. For example, in $\exists y \in \mathbf{Z}, y < x$ the variable y is (existentially) quantified, but x is not quantified.

A formula that contains only quantified variables is called a *sentence*. It represents a statement (that may be true or false).

For example, $\forall x \in \mathbf{Z}, \exists y \in \mathbf{Z}, y < x$ is a sentence, but the subformula $\exists y \in \mathbf{Z}, y < x$ is not.

The above definition of the syntax of predicate logic formulas given does not ensure that expressions are "correctly typed." For instance, if $Prime(x)$ denotes that "x is a prime number," then the expression $Prime(Socartes)$ would not make sense.

Examples

- *All men are mortal.*

$$\forall x \in D, [Man(x) \rightarrow Mortal(x)]$$

where D denotes the domain of all human beings.

- *Some students register for CSE 215.*

$$\exists x \in D, [Student(x) \wedge Register(x, cse\text{215})]$$

- *Students who do not satisfy the prerequisites may not register for CSE 215.*

$$\forall x \in D, [Student(x) \wedge {\sim}Prereqs(x, cse\text{215}) \rightarrow {\sim}Register(x, cse\text{215})]$$

- *Everyone student has an advisor.*

$$\forall x \in D, [Student(x) \rightarrow \exists y \in D, Advisor(y, x)$$

- *Only dogs bark.*

  Rephrase: *It barks only if it is a dog.*

  Or equivalently: *If it barks, then it is a dog.*

$$\forall x \in A, [Barks(x) \rightarrow Dog(x)]$$

where A is the set of all animals.

Substitution

Substitution of values for variables is a key operation in predicate logic. It is one way of turning a predicate into a statement. (The other is quantification.)

For example, if we substitute the integer 1 for the variable x in the formula $x < x^2$ we obtain $1 < 1^2$, which expresses a false statement. But if we substitute the integer 2 for x we obtain a true statement, $2 < 2^2$.

A common notation used for substitutions is to denote by $\alpha[x/a]$ the result of (simultaneously) replacing all (non-quantified) occurrences of x in α by a.

For example, if α is the formula $\exists y \in \mathbf{Z}, y < x$ and a is the constant 1, then $\alpha[x/a] = \exists y, y < 1$.

The Truth Set of a Predicate

If $\alpha(x)$ is a predicate with variable x, we define its *truth set* (with respect to a domain D) as the collection of all values $a \in D$ for which $\alpha(a)$ is true. The set-theoretic expression for this truth set is:

$$\{a \in D \,:\, \alpha(a)\}.$$

For example, if α is the formula $0 < x$ and D is the set of all integers, then the corresponding truth set is the set of all positive integers.

If α is the formula $\exists y \in \mathbf{Z}, 2 * y = x$ and D is the set of all integers, then the corresponding truth set is the set of all *even* integers.

Evaluation of Formulas

The definition of truth sets refers to the evaluation of formulas, or truth values. The truth values of predicate logic formulas, on the other hand, can be defined via truth sets.

Specifically, a universally quantified formula $\forall x \in D, \alpha(x)$ is said to evaluate to true if the truth set of $\alpha(x)$ is identical to the given domain D (i.e., *all* instances of α are true).

An existentially quantified formula $\exists x \in D, \alpha(x)$ is said to evaluate to true if the truth set of $\alpha(x)$ is nonempty (i.e., *some* instance of α is true).

There appears to be a circularity in these definitions, but one can formally define truth sets and truth values by mutual recursion on the structure (or size) of formulas.

Models and Countermodels

Most sentences are true for some domains but false for others.

For example, take the sentence

$$\forall x \in D, \exists y \in D, y < x,$$

where $<$ denotes the less-than relation. This sentence is true for the domain of *all* integers (negative and nonnegative), but false for the domain of nonnegative integers only.

By a *model* of a sentence α we mean a domain, with a corresponding interpretation of the predicate symbols, for which α is true.

A *countermodel* is a domain for which α is false.

Satisfiability and Validity

A sentence is said to be *satisfiable* if it has a model; and *unsatisfiable*, otherwise.

In other words, a sentence is unsatisfiable if every domain is a countermodel.

Sentences that are true in all domains are called *valid*.

Here are a few examples of valid sentences:

$$\forall x \in D, [P(x) \vee {\sim}P(x)]$$
$$[\forall x \in D, P(x)] \rightarrow [\exists x \in D, P(x])$$
$$\exists x \in D, (P(x) \wedge Q(x)) \rightarrow$$
$$(\exists x \in D, P(x)) \wedge (\exists x \in D, Q(x))$$

Equivalence

Two sentences α and β are said to be *equivalent*, written $\alpha \equiv \beta$, if, and only if, they have the same models.

Here is a list of well-known predicate logic equivalences, where α and β denote formulas with a non-quantified variable x. For example, α could be the formula $P(x) \to Q(x)$.

1. (a) $\sim[\forall x \in D, \alpha] \equiv \exists x \in D, \sim\alpha$
   (b) $\sim[\exists x \in D, \alpha] \equiv \forall x \in D, \sim\alpha$
   (c) $\forall x \in D, (\alpha \wedge \beta) \equiv [\forall x \in D, \alpha] \wedge [\forall x \in D, \beta]$
   (d) $\exists x \in D, (\alpha \vee \beta) \equiv [\exists x \in D, \alpha] \vee [\exists x \in D, \beta]$
   (e) $\exists x \in D, (\alpha \to \beta) \equiv [\forall x \in D, \alpha] \to [\exists x \in D, \beta]$
2. (a) $[\forall x \in D, \forall y \in D, \alpha] \equiv [\forall y \in D, \forall x \in D, \alpha]$
   (b) $[\exists x \in D, \exists y \in D, \alpha] \equiv [\exists y \in D, \exists x \in D, \alpha]$

However, note that

$$\forall x \in D, (\alpha \vee \beta) \not\equiv [\forall x \in D, \alpha] \vee [\forall x \in D, \beta]$$

and

$$\exists x \in D, (\alpha \wedge \beta) \not\equiv [\exists x \in D, \alpha] \wedge [\exists x \in D, \beta].$$

Valid Arguments

There are various logically correct arguments involving quantified statements. These can be expressed by inference rules, one example of which is the following.

Universal Instantiation:

$$\frac{\forall x \in D, \alpha(x)}{\alpha(a)}$$

where a is a constant that denotes an element of D.

For example, the statement that

all men are mortal,

can be represented by the sentence

$$\forall x \in D, [Man(x) \to Mortal(x)].$$

We may use universal instantiation to infer

$$Man(Socrates) \to Mortal(Socrates).$$

If we also know

$$Man(Socrates)$$

we may then apply Modus Ponens to infer

$$Mortal(Socrates).$$

Variable Declarations

A declaration of a variable, $x \in D$, may be viewed as a predicate.

For instance, $2 \in \mathbf{Z}$ denotes a true statement, whereas $\frac{1}{2} \in \mathbf{Z}$ is false.

The definitions of the syntax and the semantics of predicate logic can be considerably simplified if one encodes variable declarations by predicates.

In this simplified formalism we have quantified formulas

$$\forall x, \alpha \text{ and } \exists x, \alpha$$

and variables are assumed to range over a given *universal domain* (or *universe*) U. The expression

$$\forall x \in D, \alpha$$

is considered to be an abbreviation of

$$\forall x, [x \in D \to \alpha]$$

and

$$\exists x \in D, \alpha$$

an abbreviation of

$$\exists x, [x \in D \wedge \alpha].$$

The predicate $x \in D$ effectively restricts the range of the variable x to a subdomain D of U. The encoding requires a conditional in the case of universal quantification and a conjunction in the case of existential quantification.

Dracula

Let us conclude the discussion of the logic of quantified statements by an example.

Consider the following argument with two hypotheses:

> Everyone is afraid of Dracula.
> Dracula is afraid only of me.
> ∴ I am Dracula.

Is this argument logically correct?