Logical Arguments

An *argument* (form) is a (finite) sequence of statements (forms), usually written as follows:

$$\begin{array}{l} \alpha_1 \\ \dots \\ \alpha_n \\ \therefore\ \beta \end{array}$$

We call $\alpha_1, \dots, \alpha_n$ the *premises* (or *assumptions* or *hypotheses*) and β the *conclusion*, of the argument.

Argument forms are also called *inference rules* and written as

$$\frac{\alpha_1 \cdots \alpha_n}{\beta}$$

An inference rule is said to be *valid*, or *(logically) sound*, if it is the case that, for each truth valuation, if all the premises true, then the conclusion is also true.

An argument is valid if it is based on a valid argument form.

Modus Ponens

The following argument is valid:

> If Socrates is a man, then Socrates is mortal.
> Socrates is a man.
> ∴ Socrates is mortal.

Its form corresponds to a well-known inference rule:

Modus Ponens

$$\frac{\alpha \rightarrow \beta \qquad \alpha}{\beta}$$

A similar inference rule is

Modus Tollens

$$\frac{\alpha \rightarrow \beta \qquad \sim\beta}{\sim\alpha}$$

The following argument is of this form:

> If Zeus is a man, then Zeus is mortal.
> Zeus is not mortal.
> ∴ Zeus is not a man.

Valid and Invalid Arguments

A valid argument may have a false conclusion, e.g.,

> If John Lennon was a rock star, then John Lennon had red hair.
> John Lennon was a rock star.
> ∴ John Lennon had red hair.

An argument with a true conclusion may be invalid:

> If New York is a big city, then New York has tall buildings.
> New York has tall buildings.
> ∴ New York is a big city.

The validity of an argument form can be verified by constructing a suitable truth table and checking that the conclusion is true whenever all premises are true.

The question of validity can also be expressed in terms of a tautology problem:

Theorem

An inference rule

$$\frac{\alpha_1 \cdots \alpha_n}{\beta}$$

is valid if, and only if, the conditional

$$\alpha_1 \wedge \cdots \wedge \alpha_n \to \beta$$

is a tautology.

Other Inference Rules

Generalization

$$\frac{\alpha}{\alpha \vee \beta} \quad \text{and} \quad \frac{\beta}{\alpha \vee \beta}$$

Specialization

$$\frac{\alpha \wedge \beta}{\alpha} \quad \text{and} \quad \frac{\alpha \wedge \beta}{\beta}$$

Conjunction

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta}$$

Elimination

$$\frac{\alpha \vee \beta \qquad \sim\beta}{\alpha} \quad \text{and} \quad \frac{\alpha \vee \beta \qquad \sim\alpha}{\beta}$$

Transitivity

$$\frac{\alpha \to \beta \qquad \beta \to \gamma}{\alpha \to \gamma}$$

Proof Techniques

The following inference rules represent proof techniques that are often used in practice.

Proof by Division into Cases

$$\frac{\alpha \vee \beta \qquad \alpha \to \gamma \qquad \beta \to \gamma}{\gamma}$$

For instance, if a disjunction $p \vee q$ has been derived and the goal is to prove r, then according to this inference rule it would be sufficient to derive $p \to r$ and $q \to r$.

Proof by Contradiction

$$\frac{\sim\alpha \to \bot}{\alpha}$$

This rule formalizes the idea of proof by contradiction.

The usual way to derive a conditional $\sim\alpha \to \bot$ is to assume $\sim\alpha$ and then derive $\bot$ (i.e., a contradiction). Thus, if one can derive a contradiction from $\sim\alpha$, then one may conclude that α is true.

Quine's Method

The following method can be used to determine whether a given propositional formula is a tautology, a contradiction, or a contingency.

Let α be a propositional formula.

- If α contains no variables, it can be simplified to $\top$ or $\bot$, and hence is either a tautology or a contradiction.

- If α contains a variable, then (i) select a variable, say p, (ii) simplify both $\alpha[p \mapsto \top]$ and $\alpha[p \mapsto \bot]$, denoting the simplified formulas by α_1 and α_2, respectively, and (iii) apply the method recursively to α_1 and α_2.

  If α_1 and α_2 are both tautologies, so is α. If α_1 and α_2 are both contradictions, so is α. In all other cases, α is a contingency.

Example

Let α be the formula

$$(p \wedge \sim q \to r) \wedge (r \to p \vee q) \wedge (p \to \sim r) \wedge (p \vee q \vee r) \to q.$$

We first select a variable, say q, and then consider the two cases, $q = \top$ and $q = \bot$.

1) The formula $\alpha[q \mapsto \top]$ is of the form $\beta \to \top$, and hence can be simplified to $\top$.

2) We next simplify $\alpha[q \mapsto \bot]$:

$$\begin{aligned}
&(p \wedge \sim\bot \to r) \wedge (r \to p \vee \bot) \wedge (p \to \sim r) \wedge (p \vee \bot \vee r) \to \bot \\
&\quad \equiv \quad (p \wedge \top \to r) \wedge (r \to p) \wedge (p \to \sim r) \wedge (p \vee r) \to \bot \\
&\quad \equiv \quad (p \to r) \wedge (r \to p) \wedge (p \to \sim r) \wedge (p \vee r) \to \bot \\
&\quad \equiv \quad \sim[(p \to r) \wedge (r \to p) \wedge (p \to \sim r) \wedge (p \vee r)]
\end{aligned}$$

Denote the simplified formula by α_1.

We select the variable p in α_1 and consider the resulting two cases.

2.1) The formula $\alpha_1[p \mapsto \top]$ can be simplified as follows:

$$\begin{aligned}
&\sim[(\top \to r) \wedge (r \to \top) \wedge (\top \to \sim r) \wedge (\top \vee r)] \\
&\quad \equiv \quad \sim[r \wedge \top \wedge \sim r \wedge \top] \\
&\quad \equiv \quad \sim[r \wedge \sim r] \\
&\quad \equiv \quad \sim\bot \\
&\quad \equiv \quad \top
\end{aligned}$$

2.2) The formula $\alpha_1[p \mapsto \bot]$ can be simplified as follows:

$$\begin{aligned}
&{\sim}[(\bot \to r) \wedge (r \to \bot) \wedge (\bot \to {\sim}r) \wedge (\bot \vee r)] \\
&\quad\equiv \quad {\sim}[\top \wedge {\sim}r \wedge \top \wedge r] \\
&\quad\equiv \quad {\sim}[{\sim}r \wedge r] \\
&\quad\equiv \quad {\sim}\bot \\
&\quad\equiv \quad \top
\end{aligned}$$

This completes the process. All formulas considered, including the original formula α, are tautologies.

There are different ways of simplifying a formula. We assume that simplification eliminates all occurrences of $\top$ and $\bot$ from a formula. Additional simplification steps, such as replacing a subformula $p \wedge {\sim}p$ by $\bot$, are optional.