# Sequences

- A *sequence* is formally a function the domain of which is either the set of all integers greater or equal to some fixed value $s$ (an *infinite* sequence) or else the set of all integers between fixed values $s$ and $t$ (a *finite* sequence).

- For example, the function $f$ that maps each natural number $n$ to a rational number, $f(n) = (-1)^n/(n+1)$, defines an infinite sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$$

- The function $f : \{0, 1, 2, 3, 4, 5\} \to \mathbf{N}$, with

$$\begin{aligned} f(0) &= 2 \\ f(1) &= 3 \\ f(2) &= 5 \\ f(3) &= 7 \\ f(4) &= 11 \\ f(5) &= 13 \end{aligned}$$

  defines a finite sequence, usually written as

$$2, 3, 5, 7, 11, 13.$$

- Arrays are essentially finite sequences, though as a data type they also come with operations for accessing and changing array elements, etc.

# Recurrence Relations

- An infinite sequence $a$ can be defined recursively by a *recurrence relation* that specifies $a_k$ in terms of its $i$ predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where $i$ is a fixed integer (also called the *order* of the recurrence relation).

- The first $i$ values in the sequence,

$$a_s, \ldots a_{s+i-1},$$

  do not have $i$ predecessors and need to be defined separately by so-called *initial conditions*.

- For example, the function $M$ is defined by a suitable combination of a recurrence relation and initial condition:

$$\begin{aligned} M_0 &= 0 \\ M_n &= 2M_{n-1} + 1 \quad \text{if } n > 0 \end{aligned}$$

- The Fibonacci numbers are characterized by a *second-order* recurrence relation:

$$\begin{aligned} F_1 &= 1 \\ F_0 &= 1 \\ F_k &= F_{k-1} + F_{k-2} \quad \text{if } k \geq 2 \end{aligned}$$

# Summation

- If $a$ is an infinite sequence of numbers

$$a_1, a_2, a_3, \ldots$$

  we denote by $sum(1, k, a)$ the sum of the first $k$ values of the sequence $a$. Informally,

$$sum(1, k, a) = a_1 + \cdots + a_k.$$

- A precise definition of this summation function requires recursion:

$$\begin{array}{rcll} sum(1,1,a) & = & a_1 & \\ sum(1,k,a) & = & sum(1,k-1,a) + a_k & \text{if } k > 1 \end{array}$$

- Special notation is usually employed for summations:

$$\sum_{i=1}^{k} a_i = sum(1, k, a).$$

- Suppose the sequence $a$ is defined by $a_k = k^2$. One can use mathematical induction to prove that

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$$

  for all integers $n \geq 1$.

# Products

- If $a$ is an infinite sequence of numbers

  $$a_1, a_2, a_3, \ldots$$

  we denote by $prod(1, k, a)$ the product of the first $k$ values of the sequence $a$. Informally,

  $$sum(1, k, a) = a_1 * \cdots * a_k.$$

- This product function can be defined recursively, in a similar way as summation:

  $$\begin{array}{rcl} prod(1, 1, a) & = & a_1 \\ prod(1, k, a) & = & prod(1, k-1, a) * a_k \quad \text{if } k > 1 \end{array}$$

- Special notation is again employed:

  $$\prod_{i=1}^{k} a_i = prod(1, k, a).$$

# Arithmetic Sequences

- A sequence $a = a_0, a_1, \ldots$ of numbers is called an *arithmetic sequence* if there is a constant $d$ such that
$$a_k = a_{k-1} + d,$$
for all $k \geq 1$.

  In other words, the difference between two successive terms in the sequence is a constant value (the *increment*).

- An arithmetic sequence is completely determined by its initial value $a_0$ and the increment $d$. More specifically, one can prove that
$$a_n = a_0 + d * n,$$
for all $n \geq 0$.

- The latter identity may be viewed as a *solution* of the recurrence relation defining $a$.

- Proving that a proposed solution to a recurrence relation is correct typically requires a mathematical induction argument.

# Geometric Sequences

- A sequence $a$ of numbers is called a *geometric sequence* if there is a constant $r$ such that

  $$a_k = r * a_{k-1},$$

  for all $k \geq 1$.

  In other words, the ratio between any two successive terms in the sequence is constant.

- For example, the balances in an interest-bearing account in which interest is compounded (and no other deposits or withdrawals are made) will follow a geometric sequence.

- A geometric sequence is completely determined by its initial value $a_0$ and the constant $r$, and one can prove (by mathematical induction) that

  $$a_n = a_0 * r^n,$$

  for all $n \geq 0$.

# Second-Order Recurrences

- A *second-order linear homogeneous* recurrence relation is of the form
$$a_k = Aa_{k-1} + Ba_{k-2},$$
where $A$ and $B$ are real numbers with $B \neq 0$.

- For example, the Fibonacci numbers are defined by such a recurrence relation, with $A = B = 1$.

- The equation
$$x^2 - Ax - B = 0$$
is called the *characteristic equation* (of the recurrence relation).

# Solutions for Recurrences

- **Lemma**

  A sequence
  $$1, t, t^2, t^3, \ldots$$
  satisfies a second-order linear homogeneous recurrence relation
  $$a_k = Aa_{k-1} + Ba_{k-2}$$
  if, and only if, $t$ is a root (or solution) of the characteristic equation, i.e.,
  $$t^2 - At - B \text{ equals } 0.$$

- **Theorem**

  If the characteristic equation
  $$x^2 - Ax - B = 0$$
  of a recurrence relation
  $$a_k = Aa_{k-1} + Ba_{k-2}$$
  has two distinct roots $r$ and $s$, then
  $$a_k = Cr^k + Ds^k,$$
  for suitable values $C$ and $D$.

# Fibonacci Numbers

- Take the recurrence relation for the Fibonacci numbers,

$$F_k = F_{k-1} + F_{k-2}.$$

- Its characteristic equation,

$$x^2 - x - 1 = 0$$

has two distinct roots,

$$\begin{aligned} r &= (1+\sqrt{5})/2 \\ s &= (1-\sqrt{5})/2 \end{aligned}$$

- Thus we obtain an explicit formula,

$$F_k = C((1+\sqrt{5})/2)^k + D((1-\sqrt{5})/2)^k.$$

- The initial conditions $F_0 = F_1 = 1$ can be used to obtain the following equations in $C$ and $D$:

$$\begin{aligned} F_0 &= C + D &= 1 \\ F_1 &= C(1+\sqrt{5})/2 + D(1-\sqrt{5})/2 &= 1 \end{aligned}$$

- Solving for $C$ and $D$ produces the following result:

$$\begin{aligned} C &= (1+\sqrt{5})/2\sqrt{5} \\ D &= (\sqrt{5}-1)/2\sqrt{5} \end{aligned}$$

# Single Roots

- The following result applies if the characteristic equation of a second-order linear homogeneous recurrence relation has a single root.

- **Theorem**

  If the characteristic equation

  $$x^2 - Ax - B = 0$$

  of a recurrence relation

  $$a_k = Aa_{k-1} + Ba_{k-2}$$

  has a single root $r$, then

  $$a_k = Cr^k + Dkr^k,$$

  for suitable values $C$ and $D$.

- The values $C$ and $D$ are determined by the initial conditions for the sequence $a$.