Sets

- Sets are the basic data structure of mathematics.

- Informally a set is a “well-defined” collection of objects.

- Well-known examples of sets from mathematics include the set of integers, the set of rational numbers, and the set of real numbers. In computer science one often deals with sets of strings (also called “formal languages”) or trees.

- While the concept of sets is deceptively simple, all mathematical objects can in principle be described or defined in terms of sets. In that sense sets are the basic building blocks for constructing mathematical (i.e., formal, abstract) objects.

Sets and Elements

- The basic concepts of set theory are the notion of a *set* and the *element relation*.

- The symbol $\in$ is commonly used to denote the element, or membership, relation, and one writes $x \in A$ to denote the proposition that x *is an element of* A (which may be true or false).

- It is customary to denote sets by capital letters, such as A, B, and C, and elements by small letters, such as x, y, and z. But this can sometimes be confusing, for as we shall see sets can themselves be elements of other sets!

Description of Sets

- A *finite* set can (in principle) be described by listing its elements.

  That is, we write

  $$\{x_1, \ldots, x_n\}$$

  to denote the set consisting of elements $x_1, \ldots, x_n$.

- A more general mechanism for describing a set (finite or infinite) is to characterize its elements via a logical property:

  > For every set A and predicate $P(x)$ in one variable there exists a set,
  >
  > $$\{x \in A \,|\, P(x)\}$$
  >
  > that consists of all elements of A for which P is true.

- The assumption that such sets exist is known as the *Principle of Comprehension*.

Examples of Sets

- The (finite) set of integers between -2 and 5:

$$\{n \in \mathbf{Z} \mid -2 < n < 5\}$$

- The (open) interval of real numbers between -2 and 5:

$$\{x \in \mathbf{R} \mid -2 < x < 5\}$$

- The (infinite) set of even integers:

$$\{n \in \mathbf{Z} \mid \exists k \in \mathbf{Z}, (n = 2k)\}$$

- From a general description it may not always be obvious what the elements of the set are:

$$\{(x, y, z) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \mid \exists n \in \mathbf{Z}, (n > 2 \wedge x^n + y^n = z^n)\}.$$

Equality

- **Definition**

  Two sets A and B are said to be *equal*, written $A = B$, if, and only if, they have the same elements.

- More formally this can be expressed in predicate logic by the formula,

$$(A = B) \leftrightarrow \forall x, (x \in A \leftrightarrow x \in B).$$

- **Examples**

$$\begin{aligned} \{1,2\} &= \{2,1\}? \\ \{1,2\} &= \{1,1,2,2,2\}? \\ \{1,2,3\} &= \{1,1,1,3\}? \end{aligned}$$

- Note that sets are *unordered* collections of objects, where the *multiplicities* of elements *don't matter*.

Subsets

- A set A is said to be a *subset* of another set B, written $A \subseteq B$, if, and only if, every element of A is also an element of B.

- In predicate logic terms:

$$(A \subseteq B) \leftrightarrow \forall x, [x \in A \rightarrow x \in B].$$

- **Examples**

$$\begin{aligned} \{1,2\} &\subseteq \{1,2,3\}? \\ \{1,1,2,2\} &\subseteq \{1,2\}? \\ \{1\} &\subseteq \{2,3,5,7\}? \end{aligned}$$

- **Lemma.** *If $A \subseteq B$ and $B \subseteq A$, then $A = B$.*

  Proof. If $A \subseteq B$ and $B \subseteq A$, then by the definition of the subset relation, every element of A is an element of B and every element of B is an element of A. This means that A and B have the same elements, hence are equal. ∎

Proper subsets

- **Definition**

  We say that A is a *proper subset* of B, written $A \subset B$, if A is a subset of B, but not equal to B:

$$A \subset B \leftrightarrow (A \subseteq B \wedge A \neq B).$$

- **Example**

$$\{1, 2\} \subset \{1, 1, 2, 2\}?$$

- Be careful about the distinction between the element relation and the subset relation.

- **Examples**

$$\begin{array}{rcl} 2 & \in & \{1, 2, 3\}? \\ \{2\} & \in & \{1, 2, 3\}? \\ 2 & \subseteq & \{1, 2, 3\}? \\ \{2\} & \subseteq & \{1, 2, 3\}? \\ \{2\} & \subseteq & \{\{1\}, \{2\}\}? \\ \{2\} & \in & \{\{1\}, \{2\}\}? \end{array}$$

The Empty Set

- Let A be any set. How many elements are there in the set $\{x \in A : x \neq x\}$?

- A set with no elements is called an *empty set*.

- **Theorem**

  If $\emptyset$ is an empty set, then $\emptyset \subseteq A$, for all sets A.

  Proof. It is vacuously true that every element of an empty set is an element of every other set A. ∎

- **Corollary.** There is at most one empty set.

  Proof. Suppose A and B are both empty sets. By the theorem above we have $A \subseteq B$ and $B \subseteq A$, and hence $A = B$. ∎

Existence of Sets

- Another postulate of formal set theory, the *Existence Axiom*, asserts that

  there exists a set,

  which by the above considerations implies that there is an (unique) empty set.

- We use the symbol $\emptyset$, or sometimes $\{\}$, to denote the empty set.

Powersets

- **Powerset Axiom**

  If A is a set, then there exists a set, called the *powerset of* A and denoted by $\mathcal{P}(A)$, whose elements are exactly all the subsets of A:

  $$\forall x, (x \in \mathcal{P}(A) \leftrightarrow x \subseteq A).$$

- **Example.** If A is the set $\{1, 2, 3\}$, then

  $$\begin{array}{rcl} \mathcal{P}(A) & = & \{\emptyset, \\ & & \ \{1\}, \{2\}, \{3\}, \\ & & \ \{1, 2\}, \{1, 3\}, \{2, 3\}, \\ & & \ \{1, 2, 3\} \\ & & \} \end{array}$$

  Do we have $1 \in \mathcal{P}(A)$, or $2 \in \mathcal{P}(A)$, or $3 \in \mathcal{P}(A)$?

  No, because $1 \neq \{1\}$, etc.

The Size of Powersets

- If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\} \neq \emptyset$.
- **Observation**

  $\mathcal{P}(A) \neq \emptyset$, for all sets A.
- If $A = \{x\}$, then $\mathcal{P}(A) = \{\emptyset, A\}$.
- If $A = \{x, y\}$, then $\mathcal{P}(A) = \{\emptyset, \{x\}, \{y\}, A\}$.
- **Lemma**

  If A is a set with n elements, then $\mathcal{P}(A)$ has 2^n elements.

More Set Operations

- Other operations for constructing sets include
  - *set union*
  - *set intersection*
  - *relative complementation* (or *set difference*)
  - *complementation*

  They are defined as follows.

- **Definition.** Let A and B be subsets of some set S.

$$\begin{aligned} A \cup B &= \{x \in S \,|\, x \in A \vee x \in B\} \\ A \cap B &= \{x \in S \,|\, x \in A \wedge x \in B\} \\ B - A &= \{x \in S \,|\, x \in B \wedge x \notin A\} \\ A^c &= \{x \in S \,|\, x \notin A\} \end{aligned}$$

  Note that set difference can be defined as follows:

$$B - A = B \cap A^c.$$

- **Example.** Let

  S be the set of real numbers,
  A the set $\{x \in \mathbf{R} \,|\, -1 < x \leq 0\}$,
  B the set $\{x \in \mathbf{R} \,|\, 0 \leq x < 1\}$.

  What are $A \cup B$, $A \cap B$, $B - A$, and A^c?

Properties of Set Operations

- **Theorem**

  1. $A \cap B \subseteq A$ and $A \cap B \subseteq B$
  2. $A \subseteq A \cup B$ and $B \subseteq A \cup B$
  3. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

  Proof (of first property).

  > Let A and B be arbitrary sets. We prove that $A \cap B$ is a subset of A. By the definition of the subset relation, it suffices to show that every element of $A \cap B$ is an element of A. Let x be an arbitrary element of $A \cap B$. By the definition of intersection, we have $x \in A$ and $x \in B$. Thus x is an element of A. ∎

Set Identities

- Review the following identities between sets and observe their similarity to equivalences in propositional logic.

  1. Set union and intersection are commutative.
  2. Set union and intersection are associative.
  3. Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  4. Double complement: $(A^c)^c = A$.
  5. Idempotency: $A \cap A = A \cup A = A$.
  6. De Morgan's Laws:

     $$(A \cup B)^c = A^c \cap B^c$$

     and

     $$(A \cap B)^c = A^c \cup B^c.$$

  7. Absorption: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$.

Distributivity

- **Theorem.** For all sets A, B, and C,

  $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

  Proof. Let A, B, and C be arbitrary sets. We show that the two sets $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ have the same elements.

  Let x be an arbitrary but fixed element. We have:

  $$\begin{array}{ll} \multicolumn{2}{l}{x \in A \cap (B \cup C)} \\ \quad \text{iff} & x \in A \text{ and } x \in B \cup C \\ \quad \text{iff} & x \in A \text{ and } (x \in B \text{ or } x \in C) \\ \quad \text{iff} & (x \in A \text{ and } x \in B) \\ & \qquad \text{or } (x \in A \text{ and } x \in C) \\ \quad \text{iff} & x \in A \cap B \text{ or } x \in A \cap C \\ \quad \text{iff} & x \in (A \cap B) \cup (A \cap C) \end{array}$$

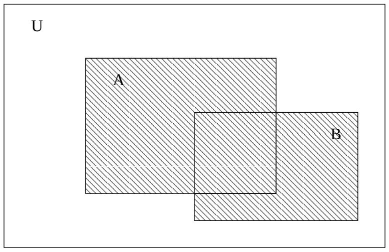
  ▮

- Note the close connection between the "algebra of sets" and the "algebra of propositions" (Boolean algebra).

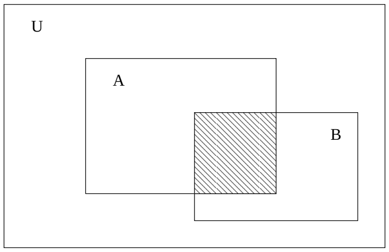
Venn Diagrams

- Sets can often be conveniently represented by *Venn diagrams*.

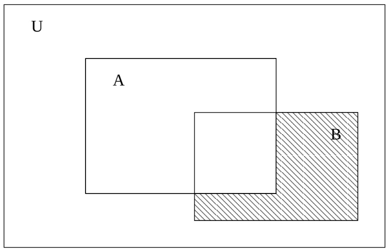
- The union $A \cup B$ of A and B is represented by:



- The intersection $A \cap B$ is represented by:



- The set difference $B - A$ is represented by:



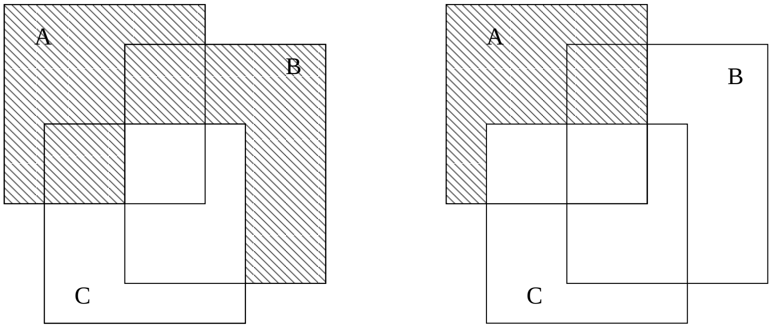
Counterexamples for Set Identities

- **Claim.** For all sets A, B, and C,

  $$(A - B) \cup (B - C) = A - C.$$

  Is this claim true?

- Consider the two Venn Diagrams:



The diagram on the left represents $(A-B)\cup(B-C)$, the one on the right, $A - C$.

The difference in the diagrams suggests a counterexample to the claim.

Take $A = \{x, y\}$, $B = \{y, z\}$, and $C = \{x, w\}$. Then $(A - B) \cup (B - C) = \{x, y, z\}$, whereas $A - C = \{y\}$.

Encoding the Natural Numbers

- If A is a set, then the set $A \cup \{A\}$ is called the *successor set* of A. The successor set of A is denoted by A'.

- The natural numbers can be encoded via successor sets:

$$\begin{array}{rcccccc}
0 & = & & & \emptyset & & \\
1 & = & 0' & = & \{\emptyset\} & = & \{0\} \\
2 & = & 1' & = & \{\emptyset, \{\emptyset\}\} & = & \{0, 1\} \\
3 & = & 2' & = & \ldots & = & \{0, 1, 2\} \\
\vdots & & & & & & \\
n + 1 & = & n' & = & \ldots & = & \{0, 1, \ldots, n\}
\end{array}$$

- In other words, we can view each natural number as an abbreviation for a certain set!

Ordered Pairs and Tuples

- Sets are *unordered* collections of elements, whereas *pairs* or *tuples* are *ordered* collections of elements.

- **Examples**

$$\begin{aligned} (1,2) &\neq (2,1) \\ \{1,2,3\} &= \{1,3,2\} \\ (1,2,3) &\neq (1,3,2) \\ \{1,2\} &= \{1,2,2\} \\ (1,2) &\neq (1,2,2) \end{aligned}$$

- Surprisingly, (ordered) pairs can be defined in terms of (unordered) sets.

- In set theory, an ordered pair (x, y) is taken as an abbreviation for the set $\{\{x\},\{x,y\}\}$.

- With this definition, do we indeed have

$$(x,y) = \{\{x\},\{x,y\}\} \neq \{\{y\},\{y,x\}\} = (y,x)?$$

  What if $x = y$?

- Tuples can be thought of as “nested” pairs. For example, we may regard $(1,2,3,4,5)$ as an abbreviation for $(1,(2,(3,(4,5))))$ or $((((1,2),3),4),5)$.

- Tuples of different length are never the same.

Cartesian Products

- Pairs and tuples provide us with a way of constructing new sets from given ones.

- If A and B are sets, then there exists a set $A \times B$ (read "A cross B"), called the *Cartesian* (or *cross*) *product* of A and B, that consists of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

- Symbolically,

$$A \times B = \{(a, b) \,|\, a \in A \wedge b \in B\}.$$

- For example, if $A = \{1, 2\}$ and $B = \{4, 5\}$, then

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5)\}.$$

- Note that A and B may be the same set.

  For instance, if $A = \{1, 3\}$, then

$$A \times A = \{(1, 1), (1, 3), (3, 1), (3, 3)\}.$$

Properties of Cartesian Products

- **Lemma**

  If A is a set of m elements and B a set of n elements, then $A \times B$ contains $m * n$ elements.

- Also note that

$$A \times \emptyset = \emptyset \times A = \emptyset.$$

- **Lemma.** For all sets A, B, and C we have

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

  Proof. We need to show that $A \times (B \cup C)$ and $(A \times B) \cup (A \times C)$ have the same elements.

$$\begin{aligned}
&(x, y) \in A \times (B \cup C) \\
&\quad \text{iff} \quad x \in A \wedge y \in B \cup C \\
&\quad \text{iff} \quad x \in A \wedge (y \in B \vee y \in C) \\
&\quad \text{iff} \quad (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\
&\quad \text{iff} \quad (x, y) \in A \times B \vee (x, y) \in A \times C \\
&\quad \text{iff} \quad (x, y) \in (A \times B) \cup (A \times C)
\end{aligned}$$

∎

Disjoint Sets

- **Definition**

  Two sets A and B are said to be *disjoint* if they have no elements in common, i.e., $A \cap B = \emptyset$.

- **Examples**

  Is $\{\emptyset, \{\emptyset\}\} \cap \{\emptyset\} = \emptyset$?
  No, $\{\emptyset, \{\emptyset\}\} \cap \{\emptyset\} = \{\emptyset\}$.
  Is $\{\emptyset, \{\emptyset\}\} \cap \emptyset = \emptyset$?
  Yes, the intersection $A \cap \emptyset$ of any set with the empty set is the empty set.

- A *partition* of a set A is a collection of pairwise disjoint nonempty sets $A_1, \ldots, A_n$, such that

$$A = A_1 \cup A_2 \cup \cdots \cup A_n.$$

- For example, at the end of the semester the set of all students taking this class will be partitioned according to final grades.

- Partitions are closely related to equivalence relations.

A Set Paradox

- Consider the set of all sets that are not elements of themselves:

  $$S = \{A \mid A \notin A\}.$$

  Is S an element of itself?

- We have

  $$S \in S \text{ if and only if } S \notin S,$$

  which is a contradiction!

- But note that the above definition of S is not covered by the Comprehension Principle. By this principle we can only define, for *some given set U*, the set

  $$S = \{A \in U \mid A \notin A\}.$$

- Now, if $S \in S$, then by the (new) definition of S, we get $S \notin S$, which would of course be a contradiction.

  Therefore we may conclude that $S \notin S$, in which case we may *also infer* $S \notin U$. We obtain no contradiction, though.

  In short, contradictions are avoided by the additional condition $A \in U$ required by comprehension.