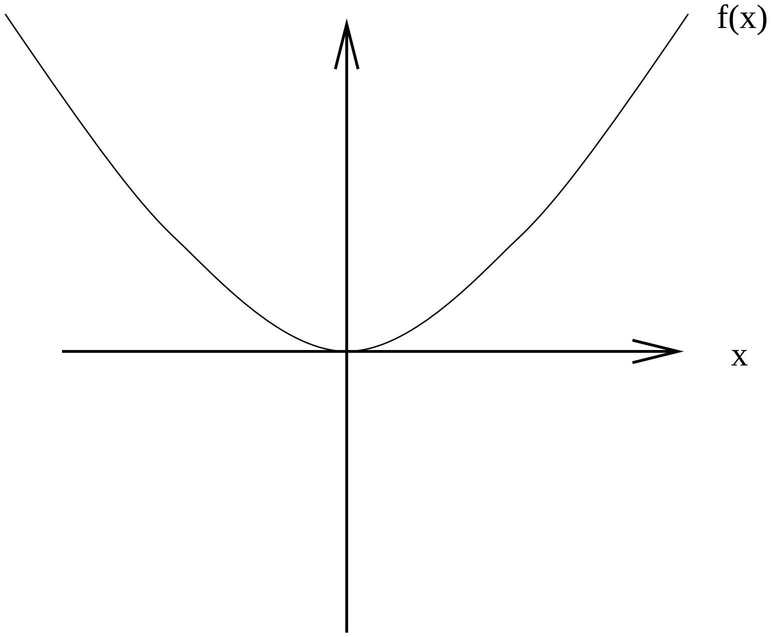
Functions - Mathematical View

- A function is a *mapping* that assigns to each element from a given domain some value from another or the same domain.

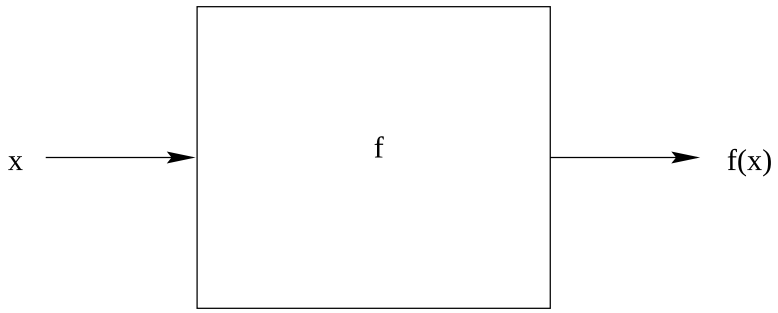
- Mathematical functions are usually drawn as graphs, e.g., the function that maps each real number x to x^2, or the function that maps x to $x^3 + x$, etc. The values along the y-coordinate are defined by an expression in terms of x, e.g., $y = x^2$ or $y = x^3 + x$.



- Note that not all identities in variables x and y define a function. For instance, the formula for a circle, $x^2 + y^2 = 1$, describes no function.

Functions - Computational View

- We can also think of a function as a (computational) *machine* that accepts (one or more) arguments as inputs and produces some value as output.



- For instance, I use a program that takes a list of exam and assignment scores as input and combines them in a certain way to produce a list of numbers (between 0 and 100) that can then be translated into letter grades. The computation reflects the guidelines announced at the beginning of the semester, e.g., that the exams contribute a fixed percentage to the total score.

Basic Terminology

- We use the notation $f : A \to B$ to indicate that f is a function from A to B and denote by $f(a)$ the value that is assigned to an argument a by f.

- For example, the *squaring function* is a function $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) = x^2$, for all real numbers x.

- A *constant function* (on the integers) is a function $f : \mathbf{Z} \to \mathbf{Z}$ such that $f(n) = k$, for all integers n, where k is a fixed value, e.g., $k = 2$.

- We call A the *domain* of the function f, and B, the *codomain*. We also speak of "a function from A to B" or, especially if $A = B$, "a function on A."

- The *range* of a function f is defined as the collection of all values y in B, such that $y = f(x)$, for some x in A.

  In the examples above range and codomain of the function are different.

Set-Theoretic Definition

- Functions are mathematical objects and can be described in terms of sets.

- In mathematics a function $f : A \rightarrow B$ is usually thought of as a subset of the cartesian product $A \times B$ that satisfies certain properties.

- In general, a set of pairs $F \subseteq A \times B$ must satisfy two conditions to specify a (total) function:

  - **Completeness**

    For each element x of the set A, there exists an element y of B, such that the pair (x, y) is in F.

  - **Uniqueness**

    F does not contain two pairs (x, y) and (x, z), where y and z are different.

- For example, the set

  $$\{(a, 1), (b, 2), (c, 1)\}$$

  describes a function from $\{a, b, c\}$ to $\{1, 2\}$, that maps a to 1, b to 2, and c to 1.

Partial Functions

- Is the (multiplicative) *inverse mapping*, which assigns to each rational number m/n the number n/m a function on the rational numbers?

  No, because n/m is not defined if $m = 0$.

  It is a function on the non-zero rational numbers, though.

- In computer science one also often encounters "partial functions," such as division, which may be "undefined" for certain arguments.

- In set-theoretic terms, a partial function is a subset of $A \times B$ that satisfies the uniqueness property, but not the completeness property.

Equality of Functions

- Two functions f and g from A to B are said to be *equal*, written $f = g$, if they agree on all arguments, i.e., $f(x) = g(x)$ for all $x \in A$.

- For example, let f and g be functions on the integers such that $f(n) = n^2 - 1$ and $g(n) = (n+1)(n-1)$. Then $f = g$.

- The *identity function* on a set A is the function $id : A \to A$ such that $id(x) = x$, for all $x \in A$.

- Is the function f on the natural numbers, such that $f(n) = n \bmod n$, equal to the identity function on $\mathbf{N}$?

Sequences

- A *sequence* is formally a function on the natural numbers, or some initial segment of the natural numbers.

- For example, the infinite sequence

  $$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$$

  can be thought of as a function f that maps each natural number n to a rational number, $f(n) = (-1)^n/(n+1)$.

- The finite sequence

  $$2, 3, 5, 7, 11, 13$$

  can be thought of as a function $f : \{0, 1, 2, 3, 4, 5\} \rightarrow \mathbf{N}$ such that

  $$\begin{array}{rcl} f(0) & = & 2 \\ f(1) & = & 3 \\ f(2) & = & 5 \\ f(3) & = & 7 \\ f(4) & = & 11 \\ f(5) & = & 13 \end{array}$$

- Arrays are essentially finite sequences, though as a data type they also come with operations for accessing array elements, changing them, etc.

Tuples

- We have seen that tuples may be thought of as “nested pairs.”
- Alternatively, a tuple may be defined as a sequence of finite length. More specifically, an n-tuple is a sequence of length n, that is, the domain of the corresponding function has n elements.
- The notation $A_1 \times A_2 \times \cdots \times A_n$ is used to denote the set of all n-tuples with the property that the first element is in A_1, the second in A_2, and so on. Sets A_1 and A_j may be different, but need not be.
- For example, $\mathbf{Z} \times \mathbf{Z}$ denotes the set of all pairs of integers.
- An example of a set of tuples with components of different type is a collection of pairs of names and corresponding ID-numbers.

Multiple-Argument Functions

- Functions of two or more arguments may be viewed as standard one-argument functions where the domain is a set of tuples (of fixed length).

- For example, a *binary* function is a function f of type $f : A_1 \times A_2 \to B$.

- **Example**. The addition function on the integers is a binary function that maps each pair of integers (m, n) to their sum $m + n$.

- In general, by an *n-ary function* we mean a function of type

  $$f : (A_1 \times A_2 \times \cdots \times A_n) \to B$$

  the domain of which is a set of n-tuples.

- It is of course also possible for the codomain of a function to be a set of pairs or tuples.

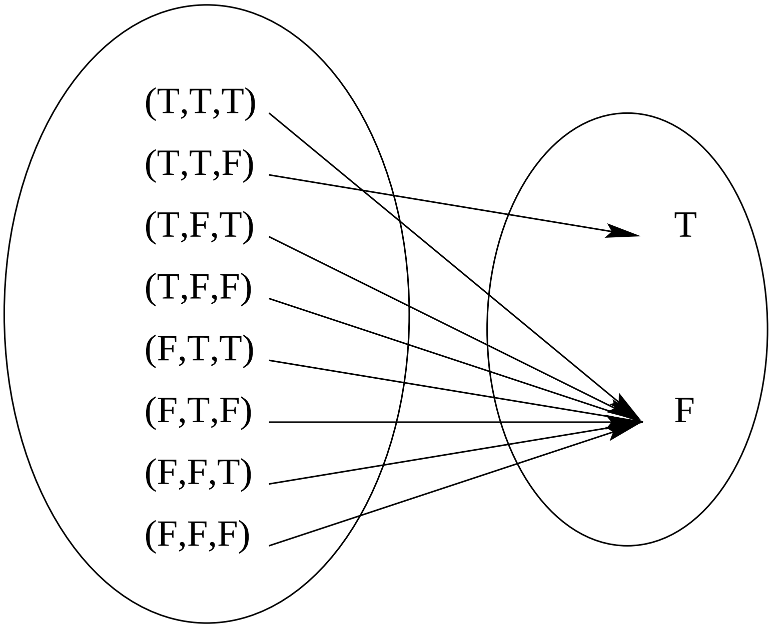
- For example, we may define a function $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ such that $f(m, n) = (q, r)$, where q and r are the quotient and remainder, respectively, of the integer division of m by n.

Boolean Functions

- Truth tables describe functions, called *Boolean functions*, that map n-tuples of truth values to single truth values.

- On the other hand, every propositional formula A defines a truth table, and hence a (unique) Boolean function.

- For example, the formula $p \wedge q \wedge \sim r$ defines the following Boolean function.



One-to-One and Onto Functions

- A function $f : X \to Y$ is said to be *one-to-one* (or *injective*) if, and only if, for all elements x and y in X, we have $x = y$ whenever $f(x) = f(y)$.

- **Example.** Let f and g be functions with domain $\{1, 2\}$ and co-domain $\{1, 2, 3\}$, defined by:

  $$\begin{array}{lcl} f(1) = 1 & \text{and} & f(2) = 1 \\ g(1) = 2 & \text{and} & g(2) = 3 \end{array}$$

  Then g is one-to-one, but f is not.

- A function $f : X \to Y$ is said to be *onto* (or *surjective*) if, and only if, for every element y in Y, there exists an element x in X, such that $f(x) = y$.

  A function is onto iff its range equals its codomain.

- For example, the function f from $\{1, 2, 3\}$ to $\{1, 2\}$, defined by:

  $$f(1) = 1,\ f(2) = 1,\ f(3) = 2$$

  is surjective, whereas no function from $\{1, 2\}$ to $\{1, 2, 3\}$ can possibly be onto.

- A function that is one-to-one and onto is said to be *bijective*.

Composition of Functions

- If the range of a function $f : X \to Y'$ is a subset of the domain Y of a function $g : Y \to Z$, we can *compose* the two functions f and g to obtain a function
$$g \circ f : X \to Z$$
defined by: $(g \circ f)(x) = g(f(x))$, for all $x \in X$.

- For example, let f be the successor function on the integers, i.e., $f(n) = n + 1$, and g be the squaring function, $g(n) = n^2$. Then
$$(f \circ g)(n) = n^2 + 1$$
whereas
$$(g \circ f)(n) = (n + 1)^2.$$

- If domain and codomain of a function are identical, it can be composed with itself:
$$(f \circ f)(n) = n + 2$$
and
$$(g \circ g)(n) = n^4.$$

Properties of Composition

- Recall that the *identity function* i_A on a domain A is defined to be a function from A to A with $i_A(x) = x$, for all elements x of the domain A.

- **Theorem**

  If f is a function from A to B, then
  $$f \circ i_A = f$$
  and
  $$i_B \circ f = f.$$

- Certain properties of functions carry over to their composition.

- **Theorem**

  If f is a one-to-one function from A to B and g a one-to-one function from B to C, then $g \circ f$ is a one-to-one function from A to C.

  If f is an onto function from A to B and g an onto function from B to C, then $g \circ f$ is an onto function from A to C.

Permutations

- One class of bijective functions are *permutations*, which rearrange a (finite) sequence of the numbers $1, \ldots, n$.

- For example, reversing a sequence is a permutation:

  $$(1,2,3,4,5,6) \mapsto (6,5,4,3,2,1)$$

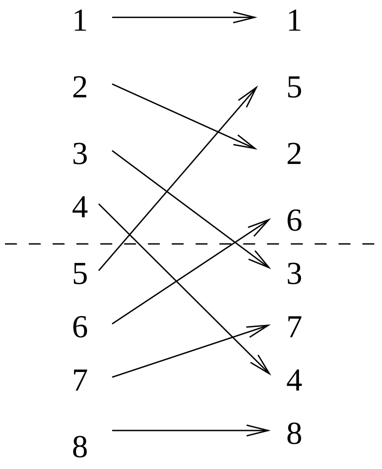
  or

  $$(3,2,6,4,1,5) \mapsto (5,1,4,6,2,3).$$

- Another examples is a *card shuffle*. We can represent the 52 cards in a deck by the numbers $1, 2, \ldots, 52$. Shuffling corresponds to rearranging a given sequence of these numbers.

- The reason we use shuffles on a deck of cards is to mix them up. A *perfect shuffle* of a deck of cards splits the deck in half and then merges the two halves so that no two adjacent cards come from the same half.

- Here is an example of a perfect shuffle on an eight card deck.



Perfect Shuffles

- What happens if we do a series of eight consecutive perfect shuffles on a 52 card deck?

- The example shows that we end up with the cards in the same order as we started!!!

- What this shows is that the composition of eight shuffles gives the identity function:

  $$x = shuf \circ shuf \circ shuf \circ shuf \circ shuf \circ shuf \circ shuf \circ shuf(x)$$

- Why do eight suffice? Observe that (except for the last card), the shuffle takes position x to position $2x \bmod 51$. Further, $2^8 \bmod 51 = 1$, so multiplying it eight times is like multiplying by 1, i.e. the identity function.

1	1	1	1	1	1	1	1	1
2	27	14	33	17	9	5	3	2
3	2	27	14	33	17	9	5	3
4	28	40	46	49	25	13	7	4
5	3	2	27	14	33	17	9	5
6	29	15	8	30	41	21	11	6
7	4	28	40	46	49	25	13	7
8	30	41	21	11	6	29	15	8
9	5	3	2	27	14	33	17	9
10	31	16	34	43	22	37	19	10
11	6	29	15	8	30	41	21	11
12	32	42	47	24	38	45	23	12
13	7	4	28	40	46	49	25	13
14	33	17	9	5	3	2	27	14
15	8	30	41	21	11	6	29	15
16	34	43	22	37	19	10	31	16
17	9	5	3	2	27	14	33	17
18	35	18	35	18	35	18	35	18
19	10	31	16	34	43	22	37	19
20	36	44	48	50	51	26	39	20
21	11	6	29	15	8	30	41	21
22	37	19	10	31	16	34	43	22
23	12	32	42	47	24	38	45	23
24	38	45	23	12	32	42	47	24
25	13	7	4	28	40	46	49	25
26	39	20	36	44	48	50	51	26
27	14	33	17	9	5	3	2	27
28	40	46	49	25	13	7	4	28
29	15	8	30	41	21	11	6	29
30	41	21	11	6	29	15	8	30
31	16	34	43	22	37	19	10	31
32	42	47	24	38	45	23	12	32
33	17	9	5	3	2	27	14	33
34	43	22	37	19	10	31	16	34
35	18	35	18	35	18	35	18	35
36	44	48	50	51	26	39	20	36
37	19	10	31	16	34	43	22	37
38	45	23	12	32	42	47	24	38
39	20	36	44	48	50	51	26	39
40	46	49	25	13	7	4	28	40
41	21	11	6	29	15	8	30	41
42	47	24	38	45	23	12	32	42
43	22	37	19	10	31	16	34	43
44	48	50	51	26	39	20	36	44
45	23	12	32	42	47	24	38	45
46	49	25	13	7	4	28	40	46
47	24	38	45	23	12	32	42	47
48	50	51	26	39	20	36	44	48
49	25	13	7	4	28	40	46	49
50	51	26	39	20	36	44	48	50
51	26	39	20	36	44	48	50	51
52	52	52	52	52	52	52	52	52

The Inverse of a Function

- If f is a bijective function from A to B, its *inverse* is a function f^{-1} from B to A, defined by:

$$f^{-1}(y) = x \text{ iff } f(x) = y.$$

- For example, if f is the function on the real numbers, defined by $f(x) = 4x - 1$, then $f^{-1}(y) = (y + 1)/4$, for all real numbers y.

- Note that the inverse of a bijective function is also bijective.

- The above definition is valid since for a bijective function f the value $f(x)$ is uniquely determined by x. But this is also the case for one-to-one functions.

  Can we define an inverse for any one-to-one function f, even if f is not onto?

  Yes, but in that case the inverse is a function from the range of f to the domain of f.

Application: Codes

- Let Σ be the set $\{a, b, c, d, e, f\}$ and Σ^* be the set of all finite sequences, or *strings*, of elements of Σ.

- Let us map characters in Σ to bitstrings as follows.

x	a	b	c	d	e	f
C(x)	000	001	010	011	100	101

- We can extend this mapping to a function on Σ^* by defining:

$$C(x_1x_2 \ldots x_n) = C(x_1)C(x_2) \ldots C(x_n).$$

- For example, the code $C(eaabb)$ is the bitstring

  100000000001001.

- Is the function C one-to-one? Onto?

- In general, encoding functions need to be one-to-one, for otherwise decoding may be difficult, or impossible.

  Character codes like this (ascii) is how text strings are represented in computers.

Encoding – Decoding

- Now consider a different, variable-length, code:

x	a	b	c	d	e	f
D(x)	0	1	10	11	100	101

- With this function we can encode $eabbe$ by a shorter bitstring

$$100011100.$$

- But can we also decode strings?

  No, the code could also represent the string $baaabbbaa$. In other words, the function is not one-to-one.

Prefix Codes

- A *prefix code function* is a function such that no codeword $C(x)$ is a prefix of another codeword $C(y)$, where x and y denote different symbols.

- Prefix code functions are one-to-one.

- Let's look at another variable-length code.

x	a	b	c	d	e	f
D'(x)	0	101	100	111	1101	1100

The encoding of $eabbe$ by D' yields

110101011011101.

Is this function one-to-one?

How are bitstrings decoded in this case?

The Pigeonhole Principle

- If A and B are finite domains and B has fewer elements than A, then there is no one-to-one function from A to B.

  This observation is also known as the *Pigeonhole Principle*.

- **Example.** Let A be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. How many of the integers from A need to be selected so that, regardless of the choice of selection, there is at least one pair with a sum of 9?

  Four is not enough, as we may select $1, 2, 3, 4$ where no pair yields a sum larger than 7.

  But any selection of five integers from A must contain a pair whose sum is 9. To see why, observe that A can be partitioned into four different subsets $A_1 = \{1, 8\}$, $A_2 = \{2, 7\}$, $A_3 = \{3, 6\}$, and $A_4 = \{4, 5\}$, where the sum of each of the four corresponding pairs is 9.

  Now if a_1, a_2, a_3, a_4, and a_5 are the selected integers from A, we define a function f, by setting $f(a_i)$ to be the set A_j that contains a_i.

  By the pigeonhole principle, the function f is not one-to-one, so that there exists two integers a_i and a_j with $f(a_i) = f(a_j)$. In other words, there must be one subset A_k, both of whose elements are selected. The corresponding sum is 9.

A Bald Statement

- Despite its simplicity, the pigeonhole principle can be used to solve an amazing variety of problems.

- *Claim:* There must be at least two non-bald New Yorkers who have exactly the same number of hairs on their heads!

- *Proof:* The maximum number of hairs on a human head is 1,000,000, and there are greater than 1,000,000 non-bald New Yorkers. ∎

- Note that this proof, although completely rigourous, is not constructive. We don't figure out which two people share the same hair count, or what the hair count is – only that the given pair must exist.

A Subset of Divisors

- Suppose you are given an arbitrary subset of 101 distinct integers from the set $S = \{1, 2, 3, \ldots, 200\}$. There must be two integers x, y in S such that x divides y.

- *Proof:* Every positive integer n can be written as $2^k \times m$, for $k \geq 0$ and m odd. (Why? Factoring all the twos from n leaves an odd number.)

  Thus every number in S can be mapped to an odd number from 1 to 199. There are exactly 100 such numbers. (Why? These are the integers $2i - 1$ for $1 \leq i \leq 100$)

  Thus at least two of the 101 distinct integers must be mapped to the same odd number m, say $x = 2^k m$ and $y = 2^{k+c} m$. Then x must divide y. ∎

  This result can be generalized to to state that any subset S of $n+1$ integers from 1 to $2n$ must contain a pair x, y in S such that x divides y.