

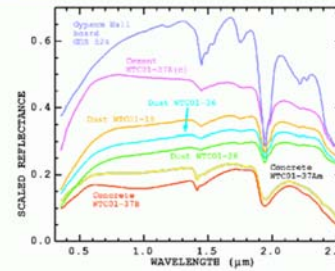
# Introduction to Medical Imaging

## Lecture 3: Linear System Theory

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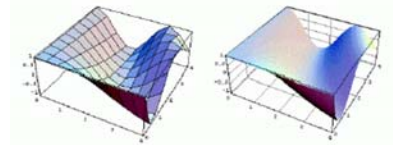
### Dimensions



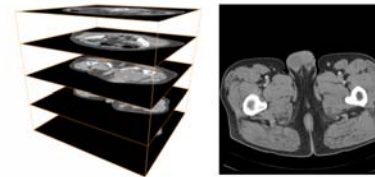
1D signal  $f(x)$



2D signal  $f(x, y)$



2D signal, shown as height field



3D signal  $f(x, y, z)$

4D signal  $f(x, y, z, t = \text{time})$   
example: 3D heart in motion

### Even / Odd Functions

Signal is even if  $s(-x) = s(x)$

- denote as  $s_e$

$$\int_{-\infty}^{+\infty} s_e(x) dx = 2 \int_0^{+\infty} s_e(x) dx$$

Signal is odd if  $s(-x) = -s(x)$

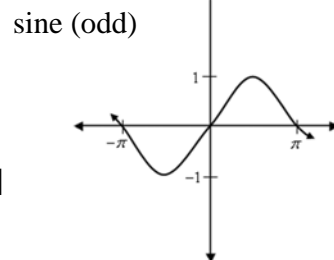
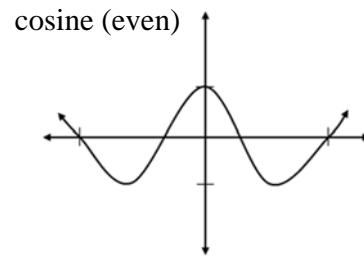
- denote as  $s_o$

$$\int_{-\infty}^{+\infty} s_o(x) dx = 0$$

Can write any signal as a sum of its even and odd part:

$$s(x) = \left[ \frac{s(x) + s(-x)}{2} \right] + \left[ \frac{s(x) - s(-x)}{2} \right]$$

$$= s_e(x) + s_o(x)$$



### Periodic Signals

A signal is periodic if  $s(x+X) = s(x)$

- we call  $X$  the period of the signal
- if there is no such  $X$  then the signal is aperiodic

Sinusoids are periodic functions

- sinusoids will play an important role in this course

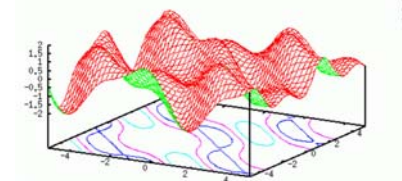
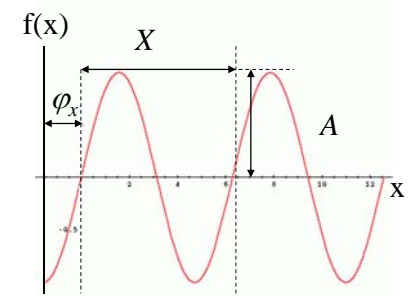
Write as:

$$A \sin\left(\frac{2\pi x}{X} + \varphi_x\right)$$

- where  $\varphi_x$  is the phase shift and  $A$  is the amplitude

Sinusoids can combine

- they can also occur in higher dimensions:



## Complex Numbers

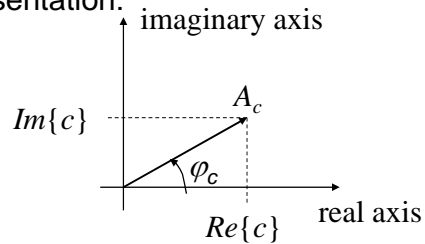
A complex number  $c$  has a real and an imaginary part:

- $c = \text{Re}\{c\} + i \text{Im}\{c\}$  (cartesian representation)  $i = \sqrt{-1}$
- here,  $i$  always denotes the complex part

We can also use a polar representation:

$$A_c = \sqrt{\text{Re}\{c\}^2 + \text{Im}\{c\}^2}$$

$$\varphi_c = \tan^{-1}\left(\frac{\text{Im}\{c\}}{\text{Re}\{c\}}\right)$$



Now think of  $c$  as a periodic signal  $s(x)$ :

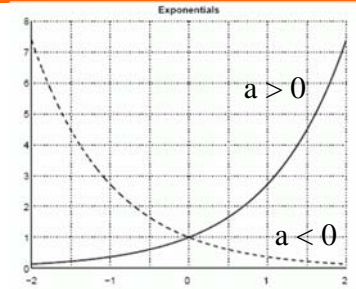
- then the pointer ( $A_c, \varphi_c$ ) rotates with period  $X$ , that is, it completes one rotation after each integer multiple of  $X$
- if there is a phase shift  $\varphi_x$  then the pointer simply is already located at ( $A, \varphi_x$ ) when  $x=0$
- considering  $c$  a 2D vector:  $\text{Re}\{c\} = A_c \cos(\varphi_c)$  and  $\text{Im}\{c\} = A_c \sin(\varphi_c)$

## Important Signals (1)

Exponential  $\exp$

$$\exp(ax) = e^{ax}$$

- when  $a > 0$  then  $\exp$  increases with increasing  $x$
- when  $a < 0$  then  $\exp$  approximates 0 with increasing  $x$

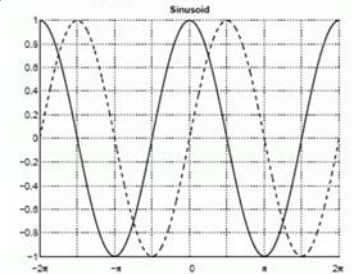


Complex exponential / sinusoid:

$$Ae^{i(2\pi kx + \phi)} = A(\cos(2\pi kx + \phi) + i \sin(2\pi kx + \phi))$$

As before

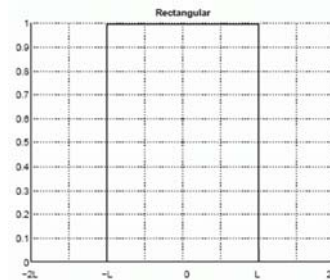
- the  $\cos$  term is the signal's real part
- the  $\sin$  term is the signal's imaginary part
- $A$  is the amplitude,  $\varphi$  the phase shift,  $k$  determines the frequency



## Important Signals (2)

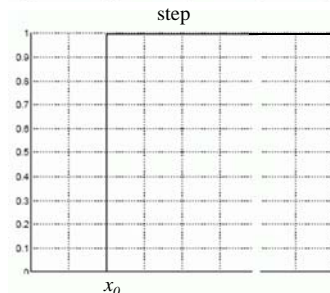
Rectangular function:

$$\begin{aligned} \Pi\left(\frac{x}{2L}\right) &= 1 \quad \text{for } |x| < L \\ &= \frac{1}{2} \quad \text{for } |x| = L \\ &= 0 \quad \text{for } |x| > L \end{aligned}$$



Step function:

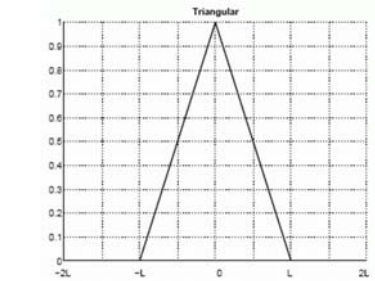
$$\begin{aligned} u(x - x_0) &= 0 \quad \text{for } x < x_0 \\ &= \frac{1}{2} \quad \text{for } x = x_0 \\ &= 1 \quad \text{for } x > x_0 \end{aligned}$$



## Important Signals (3)

Triangular function:

$$\begin{aligned} \text{Tri}\left(\frac{x}{2L}\right) &= 1 - \frac{|x|}{L} \quad \text{for } |x| < L \\ &= 0 \quad \text{for } |x| > L \end{aligned}$$

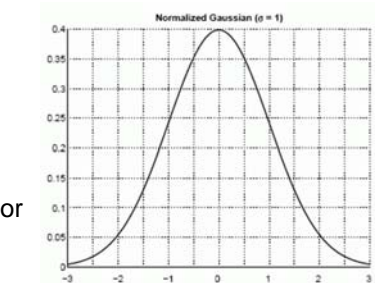


Normalized Gaussian:

$$G_n(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\mu$  is the mean  
 $\sigma$  is the standard deviation

- normalized means that the integral for all  $x$  is 0

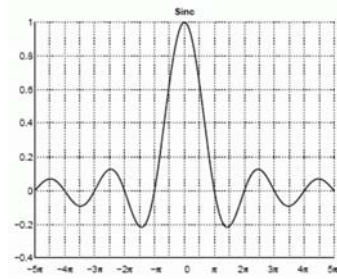


## Important Signals (4)

Sinc function:

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

- $\text{sinc}(0) = 1$  (L'Hopital's rule)



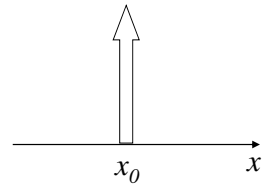
Dirac impulse:

$$\delta(x - x_0) = 0 \quad \text{for } x \neq x_0$$

$$\int_{-\infty}^{+\infty} \delta(x - x_0) dx = 1$$

- an important property is its sifting property:

$$\int_{-\infty}^{+\infty} s(x) \delta(x - x_0) dx = s(x_0)$$

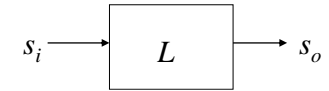


a "needle" spike of infinite height at  $x = x_0$

## Linear Systems (1)

System response  $L$ :

$$s_o = L\{s_i\}$$



- might be a function of time  $t$  or space  $x$

$$s_o(t) = L\{s_i(t)\} \quad \text{or} \quad s_o(x) = L\{s_i(x)\}$$

Finding the mathematical relationship between in- and output is called *modeling*

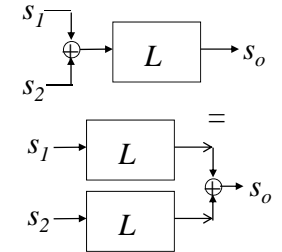
Linear systems fulfill *superposition principle*:

$$L\{c_1 s_1 + c_2 s_2\} = c_1 L\{s_1\} + c_2 L\{s_2\} \quad \forall c_1, c_2 \in \mathfrak{R}$$

where  $s_1, s_2$  are arbitrary signals

- for example, consider an amplifier with gain  $A$ :

$$\begin{aligned} L\{c_1 s_1 + c_2 s_2\} &= A(c_1 s_1 + c_2 s_2) \\ &= c_1 A s_1 + c_2 A s_2 = c_1 L\{s_1\} + c_2 L\{s_2\} \end{aligned}$$



## Linear Systems (2)

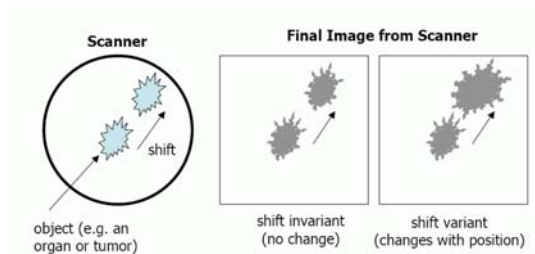
An example for a non-linear system:

$$\begin{aligned} L\{c_1 s_1 + c_2 s_2\} &= (c_1 s_1 + c_2 s_2)^2 \\ &\neq (c_1 s_1)^2 + (c_2 s_2)^2 \end{aligned}$$

Time-invariance (shift-invariance = LSI):

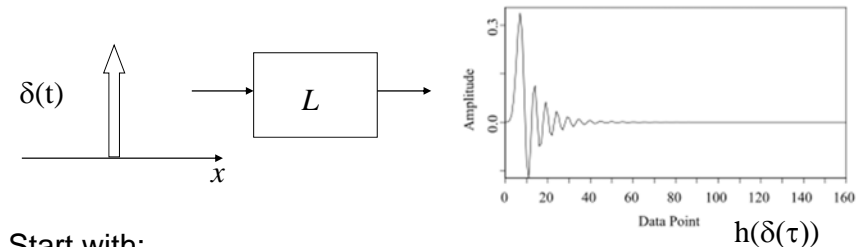
- properties of  $L$  do not change over time (spatial position), that is:

$$s_o(x) = L\{s_i(x)\} \quad \text{then} \quad s_o(x - X) = L\{s_i(x - X)\}$$



## Impulse Response (1)

A system's response to a Dirac impulse is called *impulse response*  $h$ :



Start with:

$$s_i(x) = \int_{-\infty}^{+\infty} s_i(\xi) \delta(x - \xi) d\xi$$

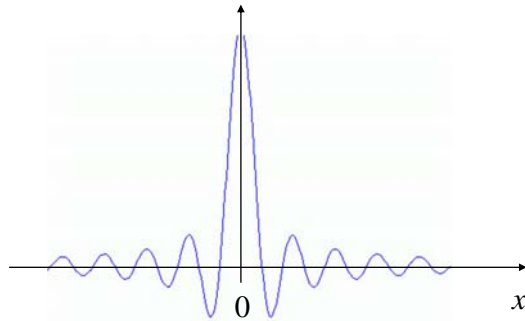
Then write:

$$s_o(x) = L\{s_i\} = \int_{-\infty}^{+\infty} s_i(\xi) L\{\delta(x - \xi)\} d\xi = \int_{-\infty}^{+\infty} s_i(\xi) h(x - \xi) d\xi$$

## Impulse Response (2)

In practice we use non-causal impulse responses

- appear symmetric in their waveform



## Convolution

The expression

$$s_o(x) = \int_{-\infty}^{+\infty} s_i(\xi)h(x-\xi)d\xi = s_i * h$$

is called *convolution*, defined as:

$$s_1(x) * s_2(x) = \int_{-\infty}^{+\infty} s_1(\xi)s_2(x-\xi)d\xi = \int_{-\infty}^{+\infty} s_1(x-\xi)s_2(\xi)d\xi$$

Procedure:

for each  $x$  do:

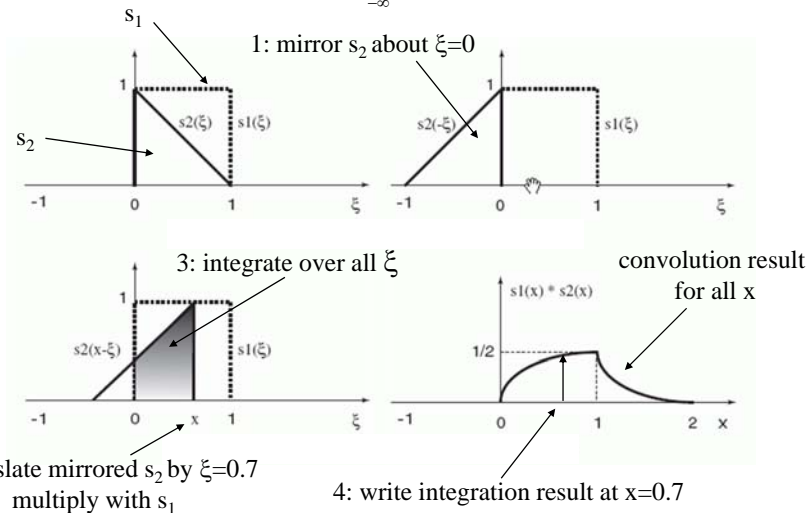
- 1: mirror  $s_2$  about  $\xi = 0$  (change  $\xi$  to  $-\xi$ )
- 2: translate mirrored  $s_2$  by  $\xi = x$
- 3: multiply  $s_1$  and mirrored  $s_2$
- 4: integrate the resulting signal

See next slides for an example and detailed explanation...

## Convolution: Example

Example  $x=0.7$ :

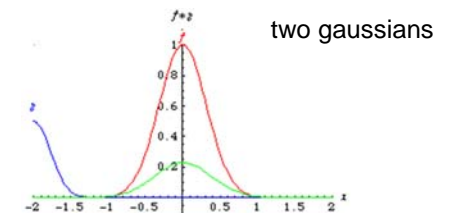
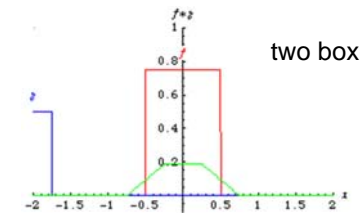
$$s_1(x) * s_2(x) = \int_{-\infty}^{+\infty} s_1(\xi)s_2(x-\xi)d\xi$$



## Convolution: More Examples

Animated gifs:

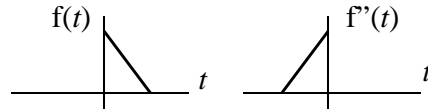
- red, blue: convolved signals
- green: convolution result



## Convolution: Detailed Explanation

Mirroring:

- when you take a function  $f(t)$  and mirror it about the y-axis then you get a new function  $f''(t) = f(-t)$



For convolution:

- you have two functions:  $f_1(t)$  and  $f_2(t)$
- you would like to compute:

$$f(x) = \int_{-\infty}^{+\infty} f_1(t)f_2(x-t)dt$$

- but in this form:  $t$  increases in  $f_1$  and decreases in  $f_2$ , which is not convenient
- to fix this, you mirror  $f_2(x-t)$  into  $f_2''(t-x) = f_2(-(x-t))$
- now the convolution writes:

$$f(x) = \int_{-\infty}^{+\infty} f_1(t)f_2''(-(x-t))dt = \int_{-\infty}^{+\infty} f_1(t)f_2''(t-x)dt$$

- at this point you need  $f_2''(t)$  which is obtained by mirroring  $f_2(t)$ :  $f_2''(t) = f_2(-t)$
- now you can do the intuitive right-sliding of  $f_2''$  for growing  $x$

## Convolution Properties

Also defined for multi-dimensional signals:

$$s_1(x, y) * s_2(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s_1(x-\xi, y-\zeta)s_2(\xi, \zeta)d\xi d\zeta$$

Some important properties:

- commutativity:

$$s_1 * s_2 = s_2 * s_1$$

- associativity:

$$(s_1 * s_2) * s_3 = s_1 * (s_2 * s_3)$$

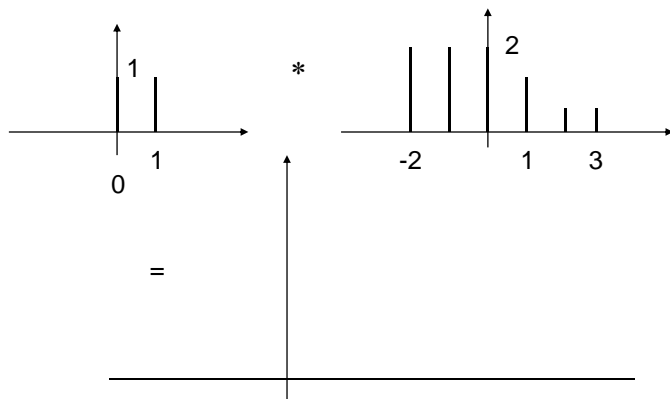
- distributivity:

$$s_1 * (s_2 + s_3) = s_1 * s_2 + s_1 * s_3$$

## Discrete Signals

Typically, signals are only available in discrete form

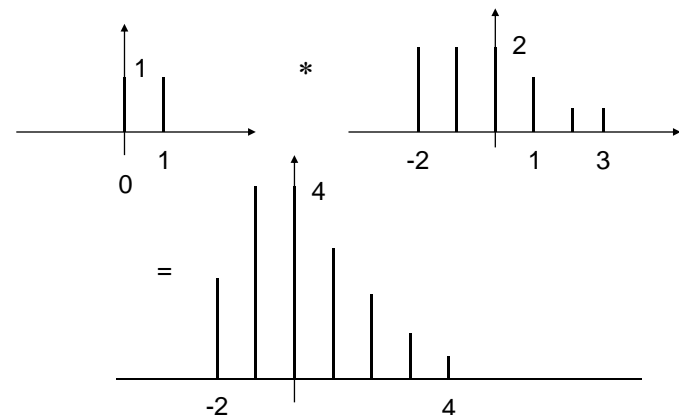
- reconstruction into a continuous signal (for visualization, etc) occurs by overlapping point spread functions (see previous lecture)
- but all computer processing (convolution and others) is done on the discrete representations



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## LSI System Response (1)

Now assume the input is a complex sinusoid with  $Ae^{2\pi i k x}$  then:

$$s_0(x) = \int_{-\infty}^{+\infty} A e^{2\pi i k(x-\xi)} h(\xi) d\xi$$

for now, assume  $\phi=0$

$$= A e^{2\pi i k x} \int_{-\infty}^{+\infty} e^{-2\pi i k \xi} h(\xi) d\xi$$

$$= A e^{2\pi i k x} H = S_i H$$

$H$  is called the *Fourier Transform* of  $h(x)$ :

$$H = \int_{-\infty}^{+\infty} e^{-2\pi i k \xi} h(\xi) d\xi$$

- $H$  is also often called the *transfer function* or *filter*
- the Fourier transform will be discussed in detail shortly

## LSI System Response (2)

$H$  scales, and maybe phase-shifts, the input sinusoid  $S_i$

In essence, we have now two alternative representations:

- determine the effect of  $L$  on  $s_i$  by convolution with  $h$ :  $s_i * h$
- determine the effect of  $L$  on  $s_i$  by multiplication with  $H$ :  $S_i \cdot H$

$$s_i * h \leftrightarrow S_i \cdot H$$

Since convolution is expensive for wide  $h$ , the multiplication may be cheaper

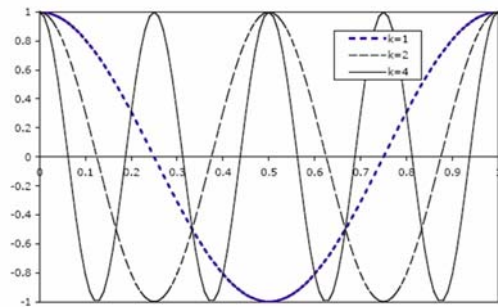
- but we need to perform the Fourier transforms of  $s_i$  and  $h$
- in fact, there is a “sweetspot”
- more later...

## Complex Sinusoids Revisited (1)

Recall the factor  $k$  in the complex sinusoid:

$$A e^{i(2\pi k x + \phi)} = A \cos(2\pi k x + \phi) + i \sin(2\pi k x + \phi)$$

- as  $k$  increases, so does the frequency of the oscillation



- note: the higher  $k$ , the higher the signal resolution, that is, one can represent smaller signal details (signals that vary more quickly)

## Signal Synthesis With Sinusoids

Any periodic signal can be created by a combination of weighted and shifted sinusoids at different frequencies

$$\begin{aligned} s_o(x) &= \int_{-\infty}^{+\infty} A_k \cos(2\pi k x + \phi_k) + i \sin(2\pi k x + \phi_k) dk \\ &= \int_{-\infty}^{+\infty} A_k e^{i(2\pi k x + \phi_k)} dk = \int_{-\infty}^{+\infty} A_k e^{i\phi_k} e^{i2\pi k x} dk \\ &= \int_{-\infty}^{+\infty} S_i(k) e^{2\pi i k x} dk \end{aligned}$$

- $A_k$  is the amplitude and  $\phi_k$  is the phase shift

Incorporating the transfer function, now one for each  $k$ :

$$s_0(x) = \int_{-\infty}^{+\infty} S_i(k) e^{2\pi i k x} H(k) dk$$

