# CSE 548: Analysis of Algorithms

# Lectures 14 & 15
# ( Dijkstra's SSSP & Fibonacci Heaps )

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# Fibonacci Heaps
## ( Fredman & Tarjan, 1984 )

A *Fibonacci heap* can be viewed as an extension of Binomial heaps which supports DECREASE-KEY and DELETE operations efficiently.

| Heap Operation | Binary Heap ( worst-case ) | Binomial Heap ( amortized ) |
|---|---|---|
| MAKE-HEAP | $\Theta(1)$ | $\Theta(1)$ |
| INSERT | $O(\log n)$ | $\Theta(1)$ |
| MINIMUM | $\Theta(1)$ | $\Theta(1)$ |
| EXTRACT-MIN | $O(\log n)$ | $O(\log n)$ |
| UNION | $\Theta(n)$ | $\Theta(1)$ |
| DECREASE-KEY | $O(\log n)$ | – |
| DELETE | $O(\log n)$ | – |

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| DECREASE-KEY | $O(\log n)$ | $O(\log n)$ ( worst case ) | $\Theta(1)$ |
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# Dijkstra's SSSP Algorithm with a Min-Heap
## ( SSSP: Single-Source Shortest Paths )

**Input:** Weighted graph $G = (V, E)$ with vertex set $V$ and edge set $E$, a weight function $w$, and a source vertex $s \in G[V]$.

**Output:** For all $v \in G[V]$, $v.d$ is set to the shortest distance from $s$ to $v$.

*Dijkstra-SSSP* ( $G = (V, E),\ w,\ s$ )

*for each* $v \in G[V]$ *do* $v.d \leftarrow \infty$
$s.d \leftarrow 0$
$H \leftarrow \phi$ { *empty min-heap* }
*for each* $v \in G[V]$ *do* INSERT( $H,\ v$ )
*while* $H \neq \emptyset$ *do*
    $u \leftarrow$ EXTRACT-MIN( $H$ )
    *for each* $v \in Adj[u]$ *do*
        *if* $v.d > u.d + w_{u,v}$ *then*
            DECREASE-KEY( $H,\ v,\ u.d + w_{u,v}$ )
            $v.d \leftarrow u.d + w_{u,v}$

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Dijkstra-SSSP ( G = (V,E), w, s )

    for each v ∈ G[V] do v.d ← ∞
    s.d ← 0
    H ← φ                                   { empty min-heap }
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    while H ≠ ∅ do
        u ← EXTRACT-MIN( H )
        for each v ∈ Adj[u] do
            if v.d > u.d + w_{u,v} then
                DECREASE-KEY( H, v, u.d + w_{u,v} )
                v.d ← u.d + w_{u,v}
```

Let $n = |G[V]|$ and $m = |G[E]|$

\# *INSERTS* $= n$
\# *EXTRACT-MINS* $= n$
\# *DECREASE-KEYS* $\leq m$

Total cost

$$\leq n(cost_{Insert} + cost_{Extract-Min}) + m(cost_{Decrease-Key})$$

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                $v.d \leftarrow u.d + w_{u,v}$

Let $n = |G[V]|$ and $m = |G[E]|$

For Binary Heap ( worst-case costs ):

$$cost_{Insert} = O(\log n)$$
$$cost_{Extract-Min} = O(\log n)$$
$$cost_{Decrease-Key} = O(\log n)$$

$\therefore$ Total cost ( worst-case )

$$= O\big((m + n) \log n\big)$$

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```

Let $n = |G[V]|$ and $m = |G[E]|$

For Binomial Heap ( amortized costs ):

$$cost_{Insert} = O(1)$$
$$cost_{Extract-Min} = O(\log n)$$
$$cost_{Decrease-Key} = O(\log n)$$

( worst-case )

∴ Total cost ( worst-case )

$$= O\big((m+n)\log n\big)$$

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Let $n = |G[V]|$ and $m = |G[E]|$

Total cost

$$\leq n(cost_{Insert} + cost_{Extract-Min}) + m(cost_{Decrease-Key})$$

**Observation:**

Obtaining a worst-case bound for a sequence of $n$ *Inserts*, $n$ *Extract-Mins* and $m$ *Decrease-Keys* is enough.

$\therefore$ Amortized bound per operation is sufficient.

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```

Let $n = |G[V]|$ and $m = |G[E]|$

Total cost

$$\leq n(cost_{Insert} + cost_{Extract-Min}) + m(cost_{Decrease-Key})$$

**Observation:**

For $n(cost_{Insert} + cost_{Extract-Min})$ the best possible bound is $\Theta(n \log n)$.
( else violates sorting lower bound )

Perhaps $m(cost_{Decrease-Key})$ can be improved to $o(m \log n)$.

# Fibonacci Heaps from Binomial Heaps

A *Fibonacci heap* can be viewed as an extension of Binomial heaps which supports DECREASE-KEY and DELETE operations efficiently.

But the trees in a Fibonacci heap are no longer binomial trees as we will be cutting subtrees out of them.

However, all operations ( except DECREASE-KEY and DELETE ) are still performed in the same way as in binomial heaps.

The *rank* of a tree is still defined as the number of children of the root, and we still link two trees if they have the same rank.

# <u>Implementing DECREASE-KEY( $\boldsymbol{H, x, k}$ )</u>

**DECREASE-KEY( $\boldsymbol{H, x, k}$ ):** One possible approach is to cut out the subtree rooted at $x$ from $H$, reduce the value of $x$ to $k$, and insert that subtree into the root list of $H$.

<u>Problem</u>: If we cut out a lot of subtrees from a tree its size will no longer be exponential in its rank. Since our analysis of EXTRACT-MIN in binomial heaps was highly dependent on this exponential relationship, that analysis will no longer hold.

<u>Solution</u>: Limit #cuts among the children of any node to 2. We will show that the size of each tree will still remain exponential in its rank.

When a 2nd child is cut from a node $x$, we also cut $x$ from its parent leading to a possible sequence of cuts moving up towards the root.

# Analysis of Fibonacci Heap Operations

Recurrence for *Fibonacci numbers*: $f_n = \begin{cases} 0 & if\ n = 0, \\ 1 & if\ n = 1, \\ f_{n-1} + f_{n-2} & otherwise. \end{cases}$

We showed in a pervious lecture: $f_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right)$,

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1+\sqrt{5}}{2}$ are the roots $z^2 - z - 1 = 0$.

# Analysis of Fibonacci Heap Operations

**Lemma 1:** For all integers $n \geq 0$, $f_{n+2} = 1 + \sum_{i=0}^{n} f_i$.

**Proof:** By induction on $n$.

Base case: $f_2 = 1 = 1 + 0 = 1 + f_0 = 1 + \sum_{i=0}^{n} f_i$.

Inductive hypothesis: $f_{k+2} = 1 + \sum_{i=0}^{k} f_i$ for $0 \leq k \leq n - 1$.

Then $f_{n+2} = f_{n+1} + f_n = f_n + \left(1 + \sum_{i=0}^{n-1} f_i\right) = 1 + \sum_{i=0}^{n} f_i$.

# Analysis of Fibonacci Heap Operations

**Lemma 2:** For all integers $n \geq 0$, $f_{n+2} \geq \phi^n$.

**Proof:** By induction on $n$.

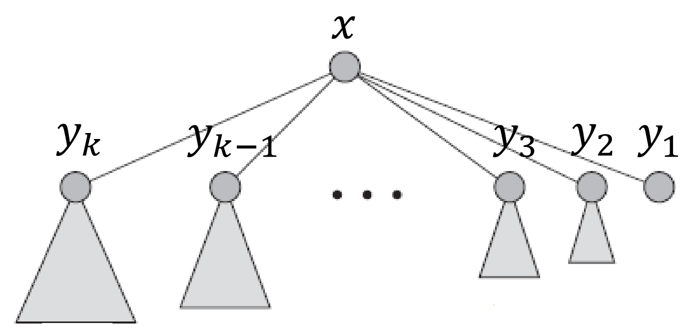
Base case: $f_2 = 1 = \phi^0$ and $f_3 = 2 > \phi^1$.

Inductive hypothesis: $f_{k+2} \geq \phi^k$ for $0 \leq k \leq n-1$.

Then
$$\begin{aligned} f_{n+2} &= f_{n+1} + f_n \\ &\geq \phi^{n-1} + \phi^{n-2} \\ &= (\phi + 1)\phi^{n-2} \\ &= \phi^2 \phi^{n-2} \\ &= \phi^n \end{aligned}$$

# Analysis of Fibonacci Heap Operations

**Lemma 3:** Let $x$ be any node in a Fibonacci heap, and suppose that $k = rank(x)$. Let $y_1, y_2, \dots, y_k$ be the children of $x$ in the order in which they were linked to $x$, from the earliest to the latest. Then $rank(y_i) \geq \max\{0, i-2\}$ for $1 \leq i \leq k$.




**Proof:** Obviously, $rank(y_1) \geq 0$.

For $i > 1$, when $y_i$ was linked to $x$, all of $y_1, y_2, \dots, y_{i-1}$ were children of $x$. So, $rank(x) \geq i-1$.

Because $y_i$ is linked to $x$ only if $rank(y_i) = rank(x)$, we must have had $rank(y_i) \geq i-1$ at that time.

Since then, $y_i$ has lost at most one child, and hence $rank(y_i) \geq i-2$.

# Analysis of Fibonacci Heap Operations

**Lemma 4:** Let $z$ be any node in a Fibonacci heap with $n = size(z)$ and $r = rank(z)$. Then $r \leq \log_{\phi} n$.

**Proof:** Let $s_k$ be the minimum possible size of any node of rank $k$ in any Fibonacci heap.

Trivially, $s_0 = 1$ and $s_1 = 2$.

Since adding children to a node cannot decrease its size, $s_k$ increases monotonically with $k$.

Let $x$ be a node in any Fibonacci heap with $rank(x) = r$ and $size(x) = s_r$.

# Analysis of Fibonacci Heap Operations

**Lemma 4:** Let $z$ be any node in a Fibonacci heap with $n = size(z)$ and $r = rank(z)$. Then $r \leq \log_\phi n$.

**Proof ( continued ):** Let $y_1, y_2, \ldots, y_r$ be the children of $x$ in the order in which they were linked to $x$, from the earliest to the latest.

Then $s_r \geq 1 + \sum_{i=1}^{r} s_{rank(y_i)} \geq 1 + \sum_{i=1}^{r} s_{\max\{0, i-2\}} = 2 + \sum_{i=2}^{r} s_{i-2}$

We now show by induction on $r$ that $s_r \geq f_{r+2}$ for all integer $r \geq 0$.

Base case: $s_0 = 1 = f_2$ and $s_1 = 2 = f_3$.

Inductive hypothesis: $s_k \geq f_{k+2}$ for $0 \leq k \leq r - 1$.

Then $s_r \geq 2 + \sum_{i=2}^{r} s_{i-2} \geq 2 + \sum_{i=2}^{r} f_i = 1 + \sum_{i=1}^{r} f_i = f_{r+2}$.

Hence $n \geq s_r \geq f_{r+2} \geq \phi^r \Rightarrow r \leq \log_\phi n$.

# Analysis of Fibonacci Heap Operations

**Corollary:** The maximum degree of any node in an $n$ node Fibonacci heap is $\mathrm{O}(\log n)$.

**Proof:** Let $z$ be any node in the heap.

Then from Lemma 4,

$$degree(z) = rank(z) \leq \log_\phi\big(size(z)\big) \leq \log_\phi n = \mathrm{O}(\log n).$$

# Analysis of Fibonacci Heap Operations

All nodes are initially unmarked.

We mark a node when

- it loses its first child

We unmark a node when

- it loses its second child, or
- becomes the child of another node ( e.g., LINKed )

We extend the potential function used for binomial heaps:

$$\Phi(D_i) = 2t(D_i) + 3m(D_i),$$

where $D_i$ is the state of the data structure after the $i^{th}$ operation,

$t(D_i)$ is the number of trees in the root list, and

$m(D_i)$ is the number of marked nodes.

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where $D_i$ is the state of the data structure after the $i^{th}$ operation,

$t(D_i)$ is the number of trees in the root list, and

$m(D_i)$ is the number of marked nodes.

**DECREASE-KEY( $\boldsymbol{H}, \boldsymbol{x}, \boldsymbol{k_x}$ ):** Let $k =$ #cascading cuts performed.

Then the actual cost of cutting the tree rooted at $x$ is 1, and

the actual cost of each of the cascading cuts is also 1.

$\therefore$ overall actual cost, $c_i = 1 + k$

# Fibonacci Heaps from Binomial Heaps

Potential function: $\Phi(D_i) = 2t(D_i) + 3m(D_i)$

**Decrease-Key( $\boldsymbol{H}, \boldsymbol{x}, \boldsymbol{k_x}$ ):**

New trees: 1 tree rooted at $x$, and

1 tree produced by each of the $k$ cascading cuts.

$\therefore t(D_i) - t(D_{i-1}) = 1 + k$

Marked nodes: 1 node unmarked by each cascading cut, and

at most 1 node marked by the last cut/cascading cut.

$\therefore m(D_i) - m(D_{i-1}) \leq -k + 1$

Potential drop, $\Delta_i = \Phi(D_i) - \Phi(D_{i-1})$

$$= 2\big(t(D_i) - t(D_{i-1})\big) + 3\big(m(D_i) - m(D_{i-1})\big)$$

$$\leq 2(1 + k) + 3(-k + 1)$$

$$= -k + 5$$

# Fibonacci Heaps from Binomial Heaps

Potential function: $\Phi(D_i) = 2t(D_i) + 3m(D_i)$

**DECREASE-KEY(** $\boldsymbol{H, x, k_x}$ **):**

$$\begin{aligned}\text{Amortized cost, } \hat{c}_i &= c_i + \Delta_i \\ &\le (1 + k) + (-k + 5) \\ &= 6 \\ &= \mathrm{O}(1)\end{aligned}$$

# Fibonacci Heaps from Binomial Heaps

Potential function: $\Phi(D_i) = 2t(D_i) + 3m(D_i)$

**EXTRACT-MIN( $H$ ):**

Let $d_n$ be the max degree of any node in an $n$-node Fibonacci heap.

Cost of creating the array of pointers is $\leq d_n + 1$.

Suppose we start with $k$ trees in the doubly linked list, and perform $l$ link operations during the conversion from linked list to array version. So we perform $k + l$ work, and end up with $k - l$ trees.

Cost of converting to the linked list version is $k - l$.

actual cost, $c_i \leq d_n + 1 + (k + l) + (k - l) = 2k + d_n + 1$

Since no node is marked, and each link reduces the #trees by 1,

potential change, $\Delta_i = \Phi(D_i) - \Phi(D_{i-1}) \geq -2l$

# Fibonacci Heaps from Binomial Heaps

Potential function: $\Phi(D_i) = 2t(D_i) + 3m(D_i)$

**EXTRACT-MIN( $H$ ):**

actual cost, $c_i \le d_n + 1 + (k + l) + (k - l) = 2k + d_n + 1$

potential change, $\Delta_i = \Phi(D_i) - \Phi(D_{i-1}) \ge -2l$

amortized cost, $\hat{c}_i = c_i + \Delta_i \le 2(k - l) + d_n + 1$

But $k - l \le d_n + 1$ ( as we have at most one tree of each rank )

So, $\hat{c}_i \le 3d_n + 3 = \mathrm{O}(\log n)$.

# Fibonacci Heaps from Binomial Heaps

Potential function: $\Phi(D_i) = 2t(D_i) + 3m(D_i)$

**DELETE( $\boldsymbol{H}, \boldsymbol{x}$ ):**

**STEP 1:** DECREASE-KEY( $H, x, -\infty$)

**STEP 2:** EXTRACT-MIN( $H$ )

amortized cost, $\hat{c}_i$ = amortized cost of DECREASE-KEY

$+$ amortized cost of EXTRACT-MIN

$= \mathrm{O}(1) + \mathrm{O}(\log n)$

$= \mathrm{O}(\log n)$