# CSE 638: Advanced Algorithms

## Lecture 8 & 9
## ( Parallel Quicksort and Selection )

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# Parallel Prefix Sums

**Input:** A sequence of $n$ elements $\{x_1, x_2, \ldots, x_n\}$ drawn from a set $S$ with a binary associative operation, denoted by $\oplus$.

**Output:** A sequence of $n$ partial sums $\{s_1, s_2, \ldots, s_n\}$, where $s_i = x_1 \oplus x_2 \oplus \ldots \oplus x_i$ for $1 \leq i \leq n$.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|
| 5 | 3 | 7 | 1 | 3 | 6 | 2 | 4 |

**$\oplus$ = binary addition**

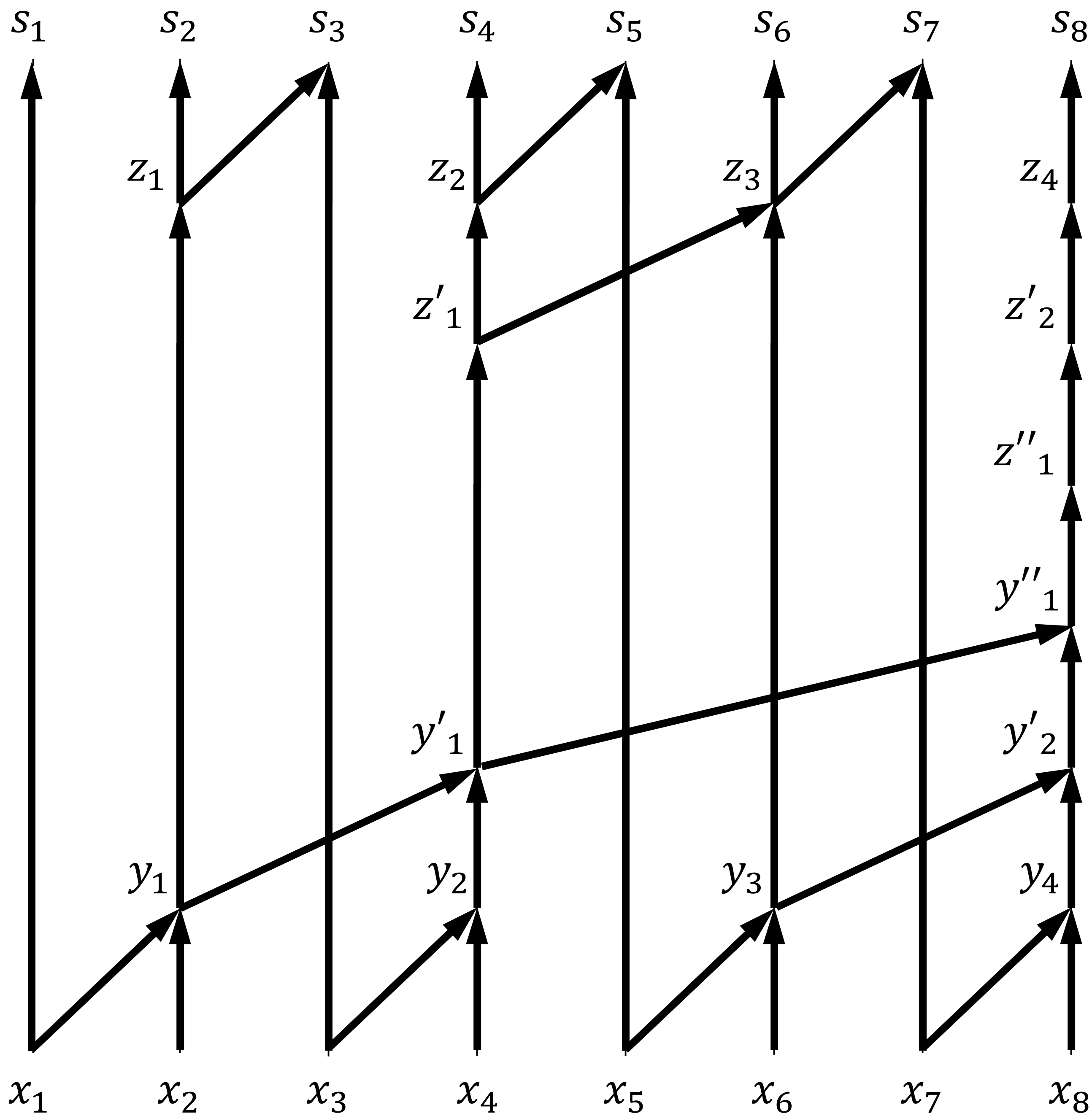
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5 | 8 | 15 | 16 | 19 | 25 | 27 | 31 |
| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | $s_8$ |

# Parallel Prefix Sums

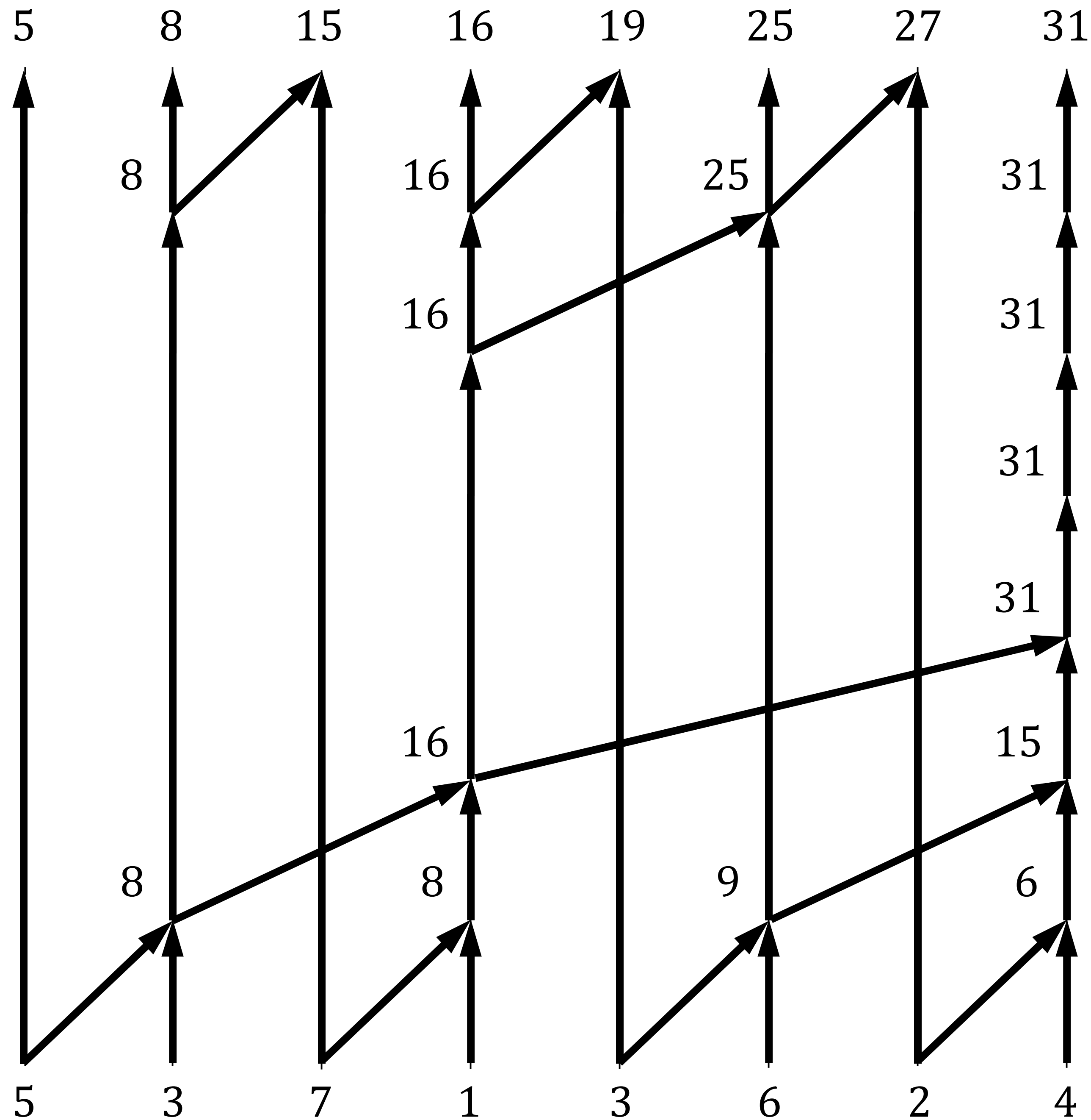
*Prefix-Sum* ( $\langle x_1, x_2, \ldots, x_n \rangle$, $\oplus$ ) { $n = 2^k$ *for some* $k \geq 0$. *Return prefix sums* $\langle s_1, s_2, \ldots, s_n \rangle$ }

*if* $n = 1$ *then*
    $s_1 \leftarrow x_1$
*else*
    *parallel for* $i \leftarrow 1$ *to* $n/2$ *do*
        $y_i \leftarrow x_{2i-1} \oplus x_{2i}$
    $\langle z_1, z_2, \ldots, z_{n/2} \rangle \leftarrow$ *Prefix-Sum*( $\langle y_1, y_2, \ldots, y_{n/2} \rangle$, $\oplus$ )
    *parallel for* $i \leftarrow 1$ *to* $n$ *do*
        *if* $i = 1$ *then* $s_1 \leftarrow x_1$
        *else if* $i = even$ *then* $s_i \leftarrow z_{i/2}$
            *else* $s_i \leftarrow z_{(i-1)/2} \oplus x_i$
*return* $\langle s_1, s_2, \ldots, s_n \rangle$

# Parallel Prefix Sums



# Parallel Prefix Sums



# Parallel Prefix Sums

```
Prefix-Sum ( ⟨x_1, x_2, ..., x_n⟩, ⊕ )    { n = 2^k for some k ≥ 0.
                                            Return prefix sums
                                            ⟨s_1, s_2, ..., s_n⟩ }
if n = 1 then
    s_1 ← x_1
else
    parallel for i ← 1 to n/2 do
        y_i ← x_{2i−1} ⊕ x_{2i}
    ⟨z_1, z_2, ..., z_{n/2}⟩ ← Prefix-Sum( ⟨y_1, y_2, ..., y_{n/2}⟩, ⊕ )
    parallel for i ← 1 to n do
        if i = 1 then s_1 ← x_1
        else if i = even then s_i ← z_{i/2}
            else s_i ← z_{(i−1)/2} ⊕ x_i
return ⟨s_1, s_2, ..., s_n⟩
```

**Work:**

$$T_1(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_1\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$$

$$= \Theta(n)$$

**Span:**

$$T_\infty(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ T_\infty\left(\frac{n}{2}\right) + \Theta(1), & otherwise. \end{cases}$$

$$= \Theta(\log n)$$

**Parallelism:** $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log n}\right)$

Observe that we have assumed here that a *parallel for loop* can be executed in $\Theta(1)$ time. But recall that *cilk_for* is implemented using divide-and-conquer, and so in practice, it will take $\Theta(\log n)$ time. In that case, we will have $T_\infty(n) = \Theta(\log^2 n)$, and parallelism $= \Theta\left(\frac{n}{\log^2 n}\right)$.

# Parallel Partition

**Input:** An array $A[\ q : r\ ]$ of distinct elements, and an element $x$ from $A[\ q : r\ ]$.

**Output:** Rearrange the elements of $A[\ q : r\ ]$, and return an index $k \in [\ q, r\ ]$, such that all elements in $A[\ q : k - 1\ ]$ are smaller than $x$, all elements in $A[\ k + 1 : r\ ]$ are larger than $x$, and $A[\ k\ ] = x$.

```
Par-Partition ( A[ q : r ], x )
n ← r − q + 1
if n = 1 then return q
array B[ 0: n − 1 ], lt[ 0: n − 1 ], gt[ 0: n − 1 ]
parallel for i ← 0 to n − 1 do
    B[ i ] ← A[ q + i ]
    if B[ i ] < x then lt[ i ] ← 1 else lt[ i ] ← 0
    if B[ i ] > x then gt[ i ] ← 1 else gt[ i ] ← 0
lt [ 0: n − 1 ] ← Par-Prefix-Sum ( lt[ 0: n − 1 ], + )
gt[ 0: n − 1 ] ← Par-Prefix-Sum ( gt[ 0: n − 1 ], + )
k ← q + lt [ n − 1 ], A[ k ] ← x
parallel for i ← 0 to n − 1 do
    if B[ i ] < x then A[ q + lt [ i ] − 1 ] ← B[ i ]
    else if B[ i ] > x then A[ k + gt [ i ] ] ← B[ i ]
return k
```

# Parallel Partition

***A*:**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 9 | 5 | 7 | 11 | 1 | 3 | 8 | 14 | 4 | 21 |

***x* = 8**

# Parallel Partition

***A***:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 9 | 5 | 7 | 11 | 1 | 3 | 8 | 14 | 4 | 21 |

***x* = 8**

***B***:

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 9 | 5 | 7 | 11 | 1 | 3 | 8 | 14 | 4 | 21 |

***lt***:

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |

***gt***:

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |

# Parallel Partition

***A***: 

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 9 | 5 | 7 | 11 | 1 | 3 | 8 | 14 | 4 | 21 |

***x* = 8**

***B***:

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 9 | 5 | 7 | 11 | 1 | 3 | 8 | 14 | 4 | 21 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| ***lt***: | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| ***lt***: | 0 | 1 | 2 | 2 | 3 | 4 | 4 | 4 | 5 | 5 |

***prefix sum***

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| ***gt***: | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| ***gt***: | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 4 |

***prefix sum***

# Parallel Partition

***A*:** $x = 8$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 9 | 5 | 7 | 11 | 1 | 3 | 8 | 14 | 4 | 21 |

***B*:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 9 | 5 | 7 | 11 | 1 | 3 | 8 | 14 | 4 | 21 |

***lt*:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |

***lt*:**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 2 | 3 | 4 | 4 | 4 | 5 | 5 |

***prefix sum***

***gt*:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |

***gt*:**

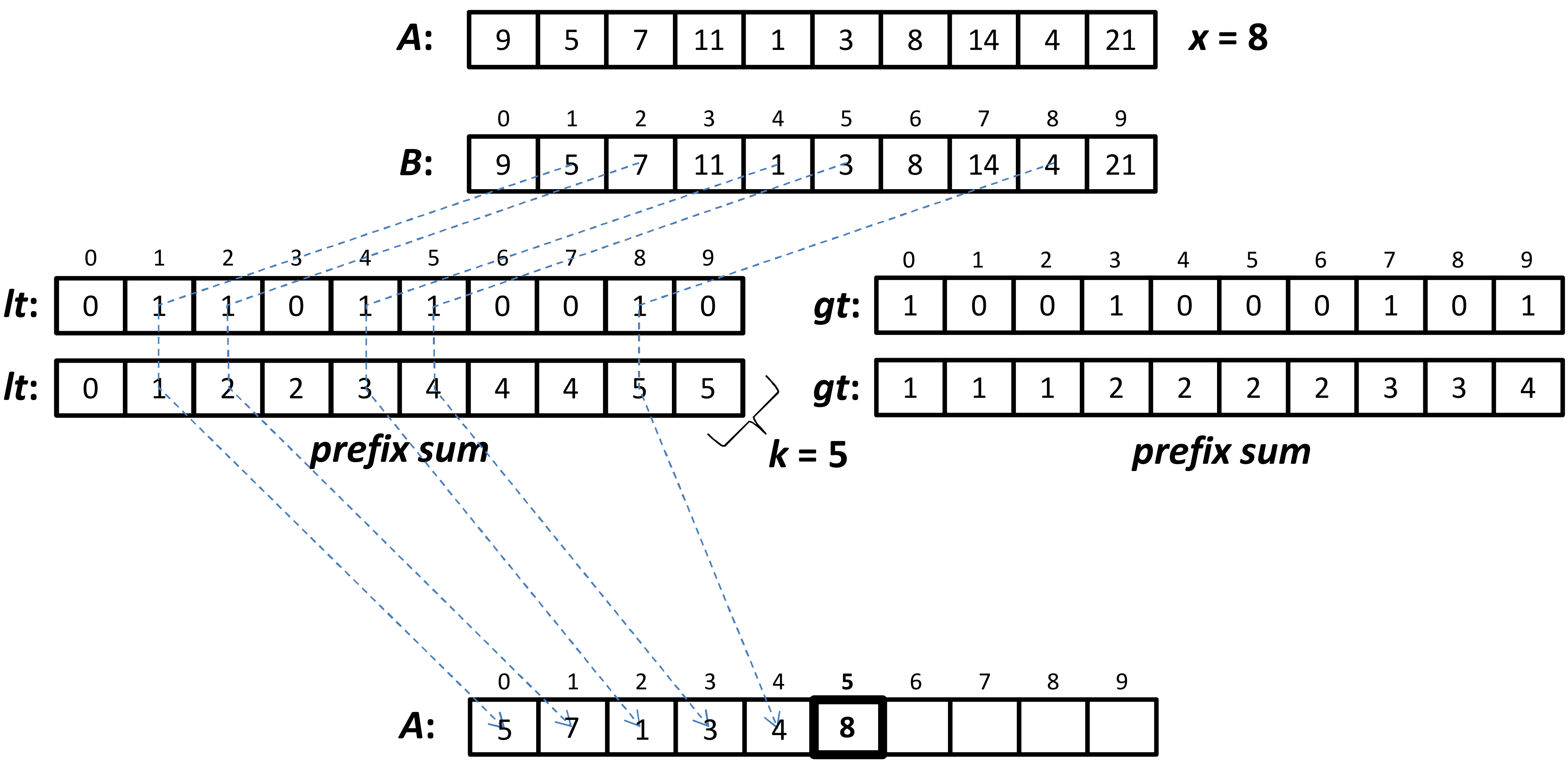
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 4 |

***prefix sum***

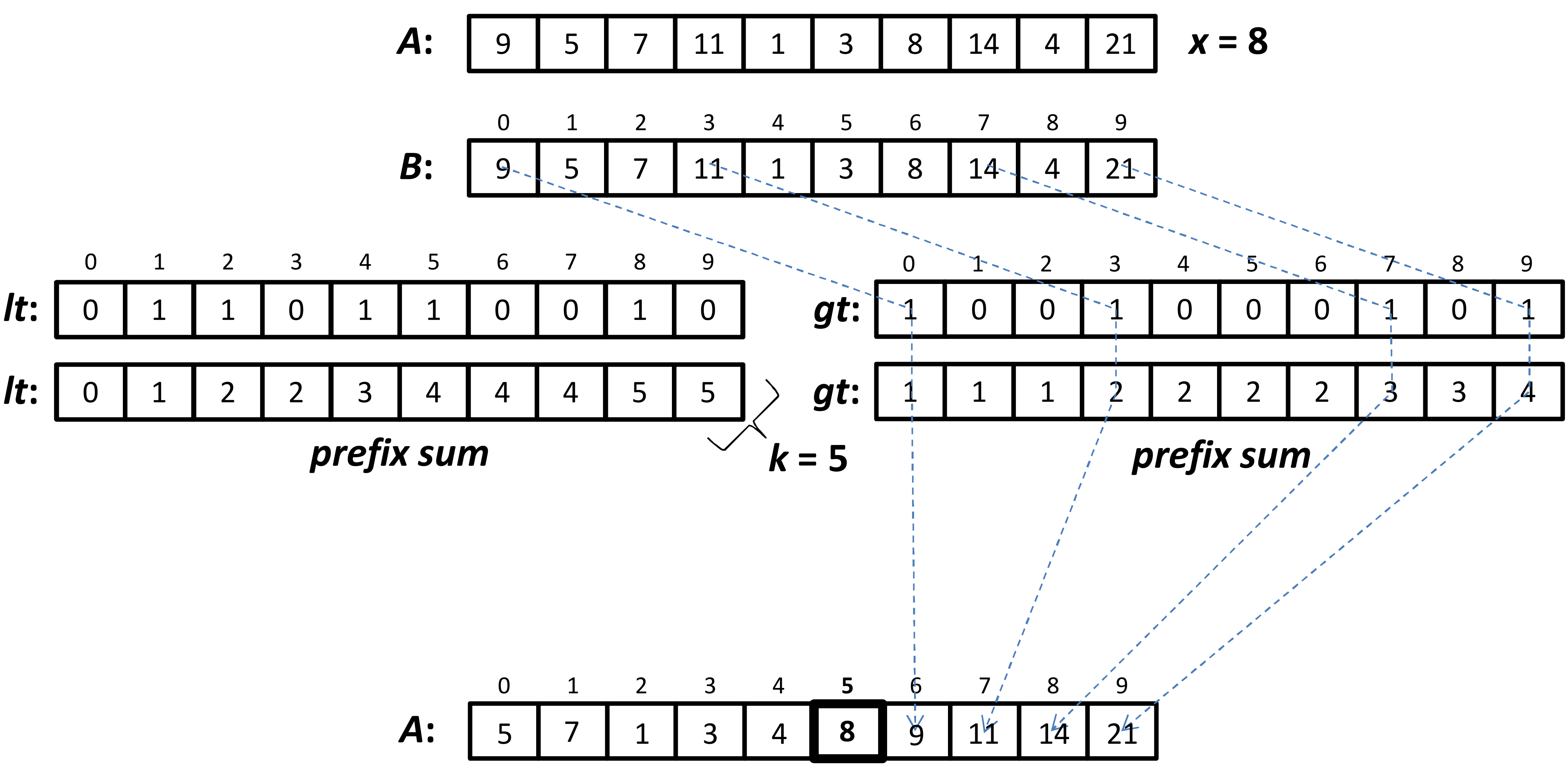
***A*:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |

# Parallel Partition



# Parallel Partition



# Parallel Partition

$x = 8$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| ***A*:** | 9 | 5 | 7 | 11 | 1 | 3 | 8 | 14 | 4 | 21 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| ***B*:** | 9 | 5 | 7 | 11 | 1 | 3 | 8 | 14 | 4 | 21 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| ***lt*:** | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| ***lt*:** | 0 | 1 | 2 | 2 | 3 | 4 | 4 | 4 | 5 | 5 |

***prefix sum***

$k = 5$

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| ***gt*:** | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| ***gt*:** | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 4 |

***prefix sum***

| | 0 | 1 | 2 | 3 | 4 | **5** | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| ***A*:** | 5 | 7 | 1 | 3 | 4 | **8** | 9 | 11 | 14 | 21 |

# Parallel Partition: Analysis

```
Par-Partition ( A[ q : r ], x )
1.  n ← r − q + 1
2.  if n = 1 then return q
3.  array B[ 0: n − 1 ], lt[ 0: n − 1 ], gt[ 0: n − 1 ]
4.  parallel for i ← 0 to n − 1 do
5.      B[ i ] ← A[ q + i ]
6.      if B[ i ] < x then lt[ i ] ← 1 else lt[ i ] ← 0
7.      if B[ i ] > x then gt[ i ] ← 1 else gt[ i ] ← 0
8.  lt [ 0: n − 1 ] ← Par-Prefix-Sum ( lt[ 0: n − 1 ], + )
9.  gt[ 0: n − 1 ] ← Par-Prefix-Sum ( gt[ 0: n − 1 ], + )
10. k ← q + lt [ n − 1 ], A[ k ] ← x
11. parallel for i ← 0 to n − 1 do
12.     if B[ i ] < x then A[ q + lt [ i ] − 1 ] ← B[ i ]
13.     else if B[ i ] > x then A[ k + gt [ i ] ] ← B[ i ]
14. return k
```

**Work:**

$$T_1(n) = \Theta(n) \quad \text{[ lines 1 – 7 ]}$$
$$+ \Theta(n) \quad \text{[ lines 8 – 9 ]}$$
$$+ \Theta(n) \quad \text{[ lines 10 – 14 ]}$$
$$= \Theta(n)$$

**Span:**

Assuming $\log n$ depth for *parallel for* loops:

$$T_\infty(n) = \Theta(\log n) \quad \text{[ lines 1 – 7 ]}$$
$$+ \Theta(\log^2 n) \quad \text{[ lines 8 – 9 ]}$$
$$+ \Theta(\log n) \quad \text{[ lines 10 – 14 ]}$$
$$= \Theta(\log^2 n)$$

**Parallelism:** $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log^2 n}\right)$

# Randomized Parallel QuickSort

**Input:** An array $A[\ q : r\ ]$ of distinct elements.

**Output:** Elements of $A[\ q : r\ ]$ sorted in increasing order of value.

```
Par-Randomized-QuickSort ( A[ q : r ] )
n ← r − q + 1
if n ≤ 30 then
    sort A[ q : r ] using any sorting algorithm
else
    select a random element x from A[ q : r ]
    k ← Par-Partition ( A[ q : r ], x )
    spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
    Par-Randomized-QuickSort ( A[ k + 1 : r ] )
    sync
```

# Randomized Parallel QuickSort: Analysis

```
Par-Randomized-QuickSort ( A[ q : r ] )
1. n ← r − q + 1
2. if n ≤ 30 then
3.      sort A[ q : r ] using any sorting algorithm
4. else
5.      select a random element x from A[ q : r ]
6.      k ← Par-Partition ( A[ q : r ], x )
7.      spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
8.      Par-Randomized-QuickSort ( A[ k + 1 : r ] )
9.      sync
```

Lines 1—6 take $\Theta(\log^2 n)$ parallel time and perform $\Theta(n)$ work.

Also the recursive spawns in lines 7—8 work on disjoint parts of $A[\ q : r\ ]$. So the upper bounds on the parallel time and the total work in each level of recursion are $\Theta(\log^2 n)$ and $\Theta(n)$, respectively.

Hence, if $D$ is the *recursion depth* of the algorithm, then

$$T_1(n) = O(nD) \text{ and } T_\infty(n) = O(D \log^2 n)$$

# Randomized Parallel QuickSort: Analysis

```
Par-Randomized-QuickSort ( A[ q : r ] )
n ← r − q + 1
if n ≤ 30 then
    sort A[ q : r ] using any sorting algorithm
else
    select a random element x from A[ q : r ]
    k ← Par-Partition (  A[ q : r ],  x )
    spawn Par-Randomized-QuickSort ( A[ q : k − 1 ] )
    Par-Randomized-QuickSort ( A[ k + 1 : r ] )
    sync
```

We will show that w.h.p. recursion depth, $D = O(\log n)$.

Hence, with high probability,

$$T_1(n) = O(n \log n) \text{ and } T_\infty(n) = O(\log^3 n)$$

# Randomized Parallel QuickSort: Analysis

**Approach:** We will show the following

1. For any specific element $v$, the sizes of the partitions containing $v$ in any two consecutive levels of recursion decrease by a constant factor with a certain probability.
2. With probability $1 - \mathrm{O}\left(\frac{1}{n^7}\right)$, the partition containing $v$ will be of size 30 or less after $\mathrm{O}(\log n)$ levels of recursion.
3. With probability $1 - \mathrm{O}\left(\frac{1}{n^6}\right)$, the partition containing every element will be of size 30 or less after $\mathrm{O}(\log n)$ levels of recursion.

# Randomized Parallel QuickSort: Analysis

**Lemma 1:** Let $v$ be an arbitrary element of the original input array $A$ of size $n = n_0$, and let $n_j$ be the size of the partition containing $v$ after partitioning at recursion depth $j \geq 1$. Then for any $j \geq 0$,

$$\Pr\left[n_{j+1} \geq \frac{7}{8}n_j\right] \leq \frac{1}{4}.$$

**Proof:** Suppose at recursion depth $j + 1 \geq 1$ element $x$ was chosen as the pivot element.

One of the new partitions will have at least $\frac{7}{8}n_j$ elements provided $x$ is among the smallest or largest $\frac{1}{8}n_j$ elements in the old partition.

The probability that $x$ is among the smallest or largest $\frac{1}{8}n_j$ elements in the old partition is clearly $\leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

# Randomized Parallel QuickSort: Analysis

**Lemma 2:** In $20 \log n$ levels of recursion, the probability that an element goes through $20 \log n - \log(n/30)$ *unsuccessful partitioning steps* (i.e., partitioning steps with $n_{j+1} < (7/8)n_j$) is $\mathrm{O}(1/n^7)$.

[ all logarithms are to the base 8/7 ]

**Proof:** The events consisting of the partitioning steps being successful can be modeled as *Bernoulli trials*.

Let $X$ be a random variable denoting the number of unsuccessful partitioning steps among the $20 \log n$ steps. Then

$$\Pr[X > 20 \log n - \log(n/30)] \leq \Pr[X > 19 \log n]$$

$$\leq \sum_{j > 19 \log n} \binom{20 \log n}{j} \left(\frac{1}{4}\right)^{j} \left(\frac{3}{4}\right)^{20 \log n - j}$$

# Randomized Parallel QuickSort: Analysis

**Lemma 2:** In $20\log n$ levels of recursion, the probability that an element goes through $20\log n - \log(n/30)$ *unsuccessful partitioning steps* (i.e., partitioning steps with $n_{j+1} < (7/8)n_j$) is $\mathrm{O}(1/n^7)$.

[ all logarithms are to the base 8/7 ]

**Proof:** $\Pr[X > 20\log n - \log(n/30)] \le \Pr[X > 19\log n]$

$$\le \sum_{j>19\log n} \binom{20\log n}{j} \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{20\log n - j}$$

$$\le \sum_{j>19\log n} \left(\frac{20e\log n}{j}\right)^j \left(\frac{1}{4}\right)^j = \sum_{j>19\log n} \left(\frac{5e\log n}{j}\right)^j$$

$$\le \sum_{j>19\log n} \left(\frac{5e\log n}{19\log n}\right)^j = \sum_{j>19\log n} \left(\frac{5e}{19}\right)^j = \mathrm{O}\left(\frac{1}{n^7}\right)$$

# Randomized Parallel QuickSort: Analysis

**Theorem 1:** The recursion depth of *Par-Randomized-Quicksort* is $\leq 20\log_{8/7} n$ with probability $1 - O(1/n^6)$.

**Proof:** The probability that one or more elements of *A* go through $20\log_{8/7} n - \log_{8/7}(n/30)$ unsuccessful partitioning steps is

$$O\left(n \times \frac{1}{n^7}\right) = O\left(\frac{1}{n^6}\right) \text{ [ using Lemma 2 ]}$$

Hence, the probability that at least $\log_{8/7}(n/30)$ of the $20\log_{8/7} n$ partitioning steps is successful for all elements is $1 - O(1/n^6)$.

After $t = \log_{8/7}(n/30)$ successful partitioning steps involving an element, the element belongs to a partition of size $\left(\frac{7}{8}\right)^t n = 30$.

Hence, *Par-Randomized-Quicksort* terminates in $\leq 20\log_{8/7} n$ levels of recursion with probability $1 - O(1/n^6)$.

# Parallel Selection

**Input:** A subarray $A[\,q : r\,]$ of an array $A[\,1 : n\,]$ of $n$ distinct elements, and a positive integer $k \in [1, r - q + 1]$.

**Output:** An element $x$ of $A[\,q : r\,]$ such that $rank(x, A[\,q : r\,]) = k$.

```
Par-Selection ( A[ q : r ], n, k )
n′ ← r − q + 1
if n′ ≤ n / log n then
     sort A[ q : r ] using a parallel sorting algorithm and return A[ q + k - 1 ]
else
     partition A[ q : r ] into blocks B_i's each containing log n consecutive elements
     parallel for i ← 1 to ⌈ n′ / log n ⌉ do
          M[ i ] ← median of B_i using a sequential selection algorithm
     find the median m of M[ 1 : ⌈ n′ / log n ⌉ ] using a parallel sorting algorithm
     t ← Par-Partition ( A[ q : r ], m )
     if k = t − q + 1 then return A[ t ]
     else if k < t − q + 1 then return Par-Selection ( A[ q : t − 1 ], n, k )
          else return Par-Selection ( A[ t + 1 : r ], n, k − t + q − 1 )
```

# Parallel Selection

**Lemma 3:** In *Par-Selection* ( lines 11—12 )

$$|A[q:t-1]| \le \frac{3n'}{4} \text{ and } |A[t+1:r]| \le \frac{3n'}{4}.$$

**Proof:** It suffices to show that $\frac{n'}{4} \le rank(m, A[q:r]) \le \frac{3n'}{4}$.

Since $m$ is the median of $M[i]$'s, it is larger than one half of the $M[i]$'s. But each $M[i]$ is larger than $\frac{\log n}{2}$ elements in $B_i$.

Hence, $rank(m, A[q:r]) \ge \frac{n'}{2 \log n} \times \frac{\log n}{2} = \frac{n'}{4}$.

Similarly, one can show that $rank(m, A[q:r]) \le \frac{3n'}{4}$.

# Parallel Selection

**Lemma 4:** In *Par-Selection* $n' \leq \frac{n}{\log n}$ after at most $\log_{4/3} \log n$ levels of recursion.

**Proof:** It follows from Lemma 3 that $n' \leq \left(\frac{3}{4}\right)^k n$ after $k$ levels of recursion.

Hence, for reaching $n' \leq \frac{n}{\log n}$, we need

$$\frac{n}{\log n} \geq \left(\frac{3}{4}\right)^k n \Rightarrow k \leq \log_{4/3} \log n.$$

# Deterministic Parallel Selection

```
Par-Selection ( A[ q : r ], n, k )
1.  n′ ← r − q + 1
2.  if n′ ≤ n / log n then
3.      sort A[ q : r ] using a parallel sorting algorithm and return A[ q + k - 1 ]
4.  else
5.      partition A[ q : r ] into blocks B_i's each containing log n consecutive elements
6.      parallel for i ← 1 to ⌈ n′ / log n ⌉ do
7.          M[ i ] ← median of B_i using a sequential selection algorithm
8.      find the median m of M[ 1 : ⌈ n′ / log n ⌉ ] using a parallel sorting algorithm
9.      t ← Par-Partition ( A[ q : r ], m )
10.     if k = t − q + 1 then return A[ t ]
11.     else if k < t − q + 1 then return Par-Selection ( A[ q : t − 1 ], n, k )
12.         else return Par-Selection ( A[ t + 1 : r ], n, k − t + q − 1 )
```

**Step 7:** Use a linear time ( worst-case ) sequential selection algorithm ( see Section 9.3 of "Introduction to Algorithms", 3rd Ed. by Cormen et al. ).

**Steps 3 and 8:** Use the parallel mergesort with parallel merge ( see Lecture 4 ) that runs in $O(\log^3 n)$ parallel time and performs $O(n \log n)$ work in the worst case.

# Deterministic Parallel Selection

```
Par-Selection ( A[ q : r ], n, k )
1.  n′ ← r − q + 1
2.  if n′ ≤ n / log n then
3.      sort A[ q : r ] using a parallel sorting
        algorithm and return A[ q + k - 1 ]
4.  else
5.      partition A[ q : r ] into blocks B_i's each
        containing log n consecutive elements
6.      parallel for i ← 1 to ⌈ n′ / log n ⌉ do
7.          M[ i ] ← median of B_i using a
            sequential selection algorithm
8.      find the median m of M[ 1 : ⌈ n′ / log n ⌉ ]
        using a parallel sorting algorithm
9.      t ← Par-Partition ( A[ q : r ], m )
10.     if k = t − q + 1 then return A[ t ]
11.     else if k < t − q + 1 then
            return Par-Selection ( A[ q : t − 1 ], n, k )
12.         else return Par-Selection ( A[ t + 1 : r ],
                                 n, k − t + q − 1 )
```

## Last Level of Recursion

**Work:** $O\left(\frac{n}{\log n}\log\left(\frac{n}{\log n}\right)\right) = O(n)$

**Span:** $O\left(\log^3\left(\frac{n}{\log n}\right)\right) = O(\log^3 n)$

## Any Other Recursion Level ( except last )

**Work:** $O\left(\frac{n'}{\log n}\times \log n\right)$ [ lines 5 – 7 ]

$+ O\left(\frac{n'}{\log n}\log\left(\frac{n'}{\log n}\right)\right)$ [ line 8 ]

$+ O(n')$ [ line 9 ]

$= O(n')$

**Span:** $O\left(\log\left(\frac{n'}{\log n}\right) + \log n\right)$ [ lines 5 – 7 ]

$+ O\left(\log^3\left(\frac{n'}{\log n}\right)\right)$ [ line 8 ]

$+ O(\log^2 n')$ [ line 9 ]

$= O(\log^3 n)$

# Deterministic Parallel Selection

```
Par-Selection ( A[ q : r ], n, k )
n′ ← r − q + 1
if n′ ≤ n / log n then
    sort A[ q : r ] using a parallel sorting
    algorithm and return A[ q + k - 1 ]
else
    partition A[ q : r ] into blocks B_i’s each
    containing log n consecutive elements
    parallel for i ← 1 to ⌈ n′ / log n ⌉ do
        M[ i ] ← median of B_i using a
        sequential selection algorithm
    find the median m of M[ 1 : ⌈ n′ / log n ⌉ ]
    using a parallel sorting algorithm
    t ← Par-Partition ( A[ q : r ], m )
    if k = t − q + 1 then return A[ t ]
    else if k < t − q + 1 then
        return Par-Selection ( A[ q : t − 1 ], n, k )
        else return Par-Selection ( A[ t + 1 : r ],
                                    n, k − t + q − 1 )
```

## Overall

**Work:**

$$T_1(n) = O\left(n + \sum_{i=0}^{\log_{4/3}\log n}\left(\frac{3}{4}\right)^i n\right)$$
$$= O(n)$$

**Span:**

$$T_\infty(n) = O\left(\left(\log_{4/3}\log n\right)\log^3 n\right)$$
$$= O(\log^3 n \log\log n)$$