

CSE150 Fall 2007 Midterm Exam Solutions

Name: \_\_\_\_\_

- You may not use any reference materials during this exam.
- Electronic devices, including calculators, cell phones, mp3 players, and laptops are all prohibited.
- You may not use your own scratch paper. The exam has plenty and you can ask for more if needed.
- You may not leave the classroom once the exam has been distributed.
- Communicating with other students in any way is prohibited.

**Academic Honesty:** I understand that if I cheat on this exam in any way, I will receive the maximum possible penalty, including an F in this course.

Name (print): \_\_\_\_\_

Signature: \_\_\_\_\_

Problem	Score
1	/25
2	/10
3	/20
4	/10
5	/10
Total	/75

## Sets, Functions, etc.

(10 points) Describe and give the cardinality of each of the following sets.

- $\emptyset \times \mathbb{Z}$

Solution:

Set:  $\emptyset$

Cardinality: 0

- $\mathbb{Z} \times \mathbb{Q}$

Solution:

Set:  $\{(z, q) : z \in \mathbb{Z} \wedge q \in \mathbb{Q}\}$

Cardinality: countably infinite, same cardinality as  $\mathbb{N}$ ,  $\aleph_0$

- $\{2^k | x \in \mathbb{Z}, 0 \leq k \leq 10\} \setminus \{2^{k+1} | x \in \mathbb{Z}, 0 \leq k \leq 10\}$

Set:  $\{2\}$

Cardinality: 1

- $\{w | w \text{ is a binary string of length 4 with two or more 0s}\} \Delta$   
 $\{w | w \text{ is a binary string of length 4 with two or more 1s}\}$

Set:  $\{1111, 1110, 1101, 1011, 0111, 0000, 0001, 0010, 0100, 1000\}$

Cardinality: 10

- $\{0, \{0, 1\}, \{0, \{1, 2\}\}\} \cap \{0, 1\}$

Set:  $\{0\}$

Cardinality: 1

(15 points) Fill in the blanks in the following table:

$A$	$B$	$f : A \rightarrow B$	$C$	$f(C)$	<i>injective</i>	<i>surjective</i>
$\mathbb{Z}$	$\mathbb{Z}$	$f(x) = x^2$	$\{2k   k \in \mathbb{Z}\}$	$\{4k^2 : k \in \mathbb{Z}\}$	<i>no</i>	<i>no</i>
$\{0, 1, 2\}$	$\{1, 2, 3\}$	$f(x) = x + 1$	$\{1, 2\}$	$\{2, 3\}$	<i>yes</i>	<i>yes</i>
$\{0, 1, \dots, 5\}$	$\{0, 1, 2, 3\}$	$f(x) = x \bmod 4$	$\{1, 5\}$	$\{1\}$	<i>no</i>	<i>yes</i>
$\mathbb{N}$	$\{0, 1\} \times \mathbb{N}$	$f(x) =$ $(x \bmod 2, \lfloor \frac{x}{2} \rfloor)$	$\{0, 1, 2, 3, 4, 5\}$	$\{(0, 0), (1, 0),$ $(0, 1), (1, 1)$ $(0, 2), (1, 2)\}$	<i>yes</i>	<i>yes</i>

## Boolean Logic

(2 points) Write down the truth table for the following function:

$$f(A, B, C) = (A \wedge B) \vee (B \wedge C) \vee (A \wedge C)$$

Solution:

$A$	$B$	$C$	$f(A, B, C)$
$T$	$T$	$T$	$T$
$T$	$T$	$F$	$T$
$T$	$F$	$T$	$T$
$T$	$F$	$F$	$F$
$F$	$T$	$T$	$T$
$F$	$T$	$F$	$F$
$F$	$F$	$T$	$F$
$F$	$F$	$F$	$F$

(8 points) Consider the following boolean operator:

$A$	$B$	$A \downarrow B$
$F$	$F$	$T$
$F$	$T$	$F$
$T$	$F$	$F$
$T$	$T$	$F$

Prove that every boolean function can be constructed from  $\downarrow$ . (Hint: show how to compute  $\neg A$  using  $\downarrow$ . Then show how to compute  $A \vee B$  using  $\neg$  and  $\downarrow$ .)

Solution:

For a boolean variable  $A$ ,  $A \downarrow A \Leftrightarrow \neg A$ , by the following table:

$A$	$A \downarrow A$
$T$	$F$
$F$	$T$

Now, since we can construct  $\neg$  from  $\downarrow$ , we can use  $\neg$  as an operator. With that, we see that  $\neg(A \downarrow B) \Leftrightarrow A \vee B$ , in the following table:

$A$	$B$	$\neg(A \downarrow B)$
$T$	$T$	$T$
$T$	$F$	$T$
$F$	$T$	$T$
$F$	$F$	$F$

As proved in class, any boolean function can be constructed using only  $\neg$  and  $\vee$ , and since we can construct both of those with only  $\downarrow$ , any boolean function can be constructed from  $\downarrow$ .

## Relations

(20 points) Let  $A$  be a set,  $R \subseteq A \times A$ ,  $T : P(A \times A) \rightarrow P(A \times A)$ ,  $S : P(A \times A) \rightarrow P(A \times A)$ ,  $T(R)$  gives the transitive closure of  $R$ , and  $S(R)$  gives the symmetric closure of  $R$ .

- Give an example  $A$  and  $R$  such that  $T(S(R)) \not\subseteq S(T(R))$

Solution:

$$A = \{1,2,3\}$$

$$R = \{(1, 2), (3, 2)\}$$

Now,  $S(R) = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$  and  $T(S(R)) = P(A)$ , but  $T(R) = R$ , so  $S(T(R)) = S(R)$ . Thus,  $T(S(R)) \not\subseteq S(T(R))$ .

- Prove that for all  $A$  and  $R$ ,  $S(T(R)) \subseteq T(S(R))$

**Proof** Suppose  $(a, b) \in S(T(R))$ . Then  $(a, b) \in T(R)$  or  $(b, a) \in T(R)$ . Consequently, there exist  $x_1, x_2, \dots, x_k$  such that

$$(a, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{k-1}, x_k), (x_k, b) \in R$$

or

$$(b, x_k), (x_k, x_{k-1}), (x_{k-1}, x_{k-2}), \dots, (x_2, x_1), (x_1, a) \in R$$

In either case,

$$(a, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{k-1}, x_k), (x_k, b) \in S(R)$$

Consequently,  $(a, b) \in T(S(R))$ .

## Induction

(10 points) Let  $x_2 = \sqrt{1+1}$  and for  $n > 2$ ,  $x_n = \sqrt{1+x_{n-1}}$ . Prove that  $x_i$  is irrational for all  $i \geq 2$ . (Note: you may assume  $\sqrt{2}$  is irrational, since we proved it in class.)

**Proof** The proof is by induction on  $n$ .

- When  $n = 2$ ,  $x_n = \sqrt{2}$ , which is known to be irrational.
- Now suppose  $x_n$  is irrational. Suppose, for the sake of contradiction, that  $x_{n+1} = \sqrt{1+x_n}$  is rational. Then  $x_{n+1}^2 - 1 = x_n$  is also rational, contradicting our inductive hypothesis. Thus  $x_{n+1}$  must be irrational.

## Number Theory

(10 points) Let  $p$  be a prime number and  $a, b \in \mathbb{Z}$ . Prove that  $(a + b)^p \bmod p = (a^p + b^p) \bmod p$ .  
Hint:  $(a + b)^p = \sum_{i=0}^p \binom{p}{i} a^i b^{p-i}$ , where  $\binom{p}{i} = \frac{p!}{(p-i)!i!} = \frac{p(p-1)(p-2)\cdots 1}{(p-i)(p-i-1)(p-i-2)\cdots 1 \cdot i(i-1)(i-2)\cdots 1}$ .

**Proof** First, we argue that when  $0 < i < p$ ,  $p \mid \binom{p}{i}$ . Recall that  $\binom{p}{i}$  is an integer for all  $i$ . Observe that, when  $0 < i < p$ , every factor in the denominator is less than  $p$ . Thus, when we cancel out all the common factors in the numerator and denominator, there will still be a factor of  $p$  left in the numerator. Since  $\binom{p}{i}$  is an integer, cancelling all the common factors in the numerator and denominator must therefore leave a fraction of the form  $pm/1 = pm$ . Hence  $p \mid \binom{p}{i}$  for  $0 < i < p$ .

Now observe that

$$\begin{aligned} (a + b)^p \bmod p &= \sum_{i=0}^p \binom{p}{i} a^i b^{p-i} \bmod p \\ &= a^p + b^p + \sum_{i=1}^{p-1} \binom{p}{i} a^i b^{p-i} \bmod p \\ &= a^p + b^p + p \left( \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} a^i b^{p-i} \right) \bmod p \\ &= a^p + b^p + p \bmod p \left( \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} a^i b^{p-i} \right) \bmod p \\ &= a^p + b^p + 0 \cdot \left( \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} a^i b^{p-i} \right) \bmod p \\ &= a^p + b^p \bmod p \end{aligned}$$