CSE 150: Problem Set #2

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Problem 1

Find sets A and B satisfying each of the following condictions. Make your sets as small as possible.

1. $|A| < |B|, A \not\subset B$
2. $|P(A)| > 13$.
3. The function $f : A \to B$, $f(x) = \sin x$ is an injection.
4. $A \neq \emptyset$, and the relation $R \subseteq A \times B$ where $xRy \iff y = \sin x$ is an equivalence relation.

Grading

One point per problem, if you got both A and B correct.

Problem 2

How would you change the definitions of the following relations to make them equivalence relations, and why?

1. $R \subseteq$ People $\times$ People. xRy iff x and y were born within 24 hours of each-other.
2. $R \subseteq$ Cities $\times$ Cities. xRy iff it is possible to fly directly from x to y.
3. $R \subseteq$ People $\times$ People. xRy iff x and y have the same mother and y is at least as old as x.

Grading

For each sub-problem:

- 1 point was given for identifying the reason R was not an equivalence relation,
- 1 point was given for giving a correct equivalence relation with the same flavor as R, and
- 1 point was given for giving proper justification for why your new relation is an equivalence relation

Problem 3

Describe and count the equivalence classes of each relation given below. Is each equivalence class the same size?

1. $R \subseteq \mathbb{Z} \times \mathbb{Z}$. xRy iff $x \bmod 7 = y \bmod 7$
2. $R \subseteq \mathbb{Z} \times \mathbb{Z}$. xRy iff $\lfloor \frac{x}{7} \rfloor = \lfloor \frac{y}{7} \rfloor$. (Definition: $\lfloor z \rfloor$ is the largest integer less than or equal to z)
3. Let $B = \{0, 1\}$, and B^n be the set of n-bit strings. $R \subseteq B^n \times B^n$. xRy iff x and y have the same number of 0s.

Grading

For each sub-problem:

- 1 point for describing the equivalence classes with a reasonable amount of precision and mathematical rigor (some points were taken off for using ellipses $(\dots)$, but not all)
- 1 point for correctly giving the number of equivalence classes
- 1 point for identifying the cardinalities of each equivalence class (or at least saying whether all were the same size or not)

Problem 4

For each function, indicate whether it is injective, surjective, both, or neither.

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \log x$.
2. $f : \mathbb{N} \to \mathbb{R}$, $f(x) = \cos \pi x$.
3. $f : \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$, $f(x) = 3x \bmod 5$.
4. $f : \{0, 1, 2, 3, 4, 5\} \to \{0, 1, 2, 3, 4, 5\}$, $f(x) = 3x \bmod 6$.

Grading

For each sub-problem:

- 1 point for correctly determining whether f was injective
- 1 point for correctly determining whether f was surjective

Problem 5

Write a formal proof that if $A \subseteq B$, then $|A| \leq |B|$, i.e. there exists a surjection $f : B \to A$.

Grading

Up to 4 points total for this problem. Points came off for some of the following reasons:

- Assuming things that are not true
- Defining a function f that was not defined on all points in B (this happened a lot)
- Excessively bad proof style (various reasons)

Problem 6

Let $B = \{0, 1\}$. For $n \in \mathbb{N}$, we call B^n the binary strings of length n. Note the $B^1 = B$ and $B^0 = \{\epsilon\}$, where ϵ is the string of length 0. Let $B^* = \cup_{n=0}^{\infty} B^n$. Note that every string in B^* has finite length.

1. Prove, by induction on n, that $|B^n| = 2^n$.
2. Prove that B^* is countable.
3. Every computer program is a binary string. What is the size of the set of all computer programs?

Grading

For sub-problems 1 and 2:

- 1 point for a correct base case
- 1 point for a correct inductive hypothesis (and following proof)
- 1 point for proof style (could be deducted for small false assumptions or other reasons)

For sub-problem 3, 1 point was available.

Problem 7

Consider a simple programming language that only has three kinds of statements:

- `y:=E`, where `E` is some expression in terms of variables, +, -, ×, and /.
- `S; T`, which executes `S` and then `T`.
- `for (i = a ... b) S`, which executes `S` once for each integer in the range $[a, b]$.

Prove that every program written in this language terminates (i.e. no program can run forever). In other words, prove that for every program P, there exists an integer N such that P always runs in N seconds or less.

Grading

4 points available, in the same manner as Problem 5.

Problem 8

- Prove that $\Sigma_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.
- How many breaks does it take to break an $m \times n$ piece of chocolate into 1×1 pieces? Prove your answer.

Grading

For each sub-problem, 2 points were available, on a mostly subjective basis (there were not enough errors common to a large set of students to create a decent map from solution-elements to grade points).

Bonus Problem

Prove that an integer is divisible by 9 iff the sum of its digits (in base 10) is divisible by 9.

Grading

`__BONUS` points maximum for this problem. I indicated whether `__BONUS` or $\frac{1}{2}$`__BONUS` points are to be awarded. I left it up to Rob to determine the correct value for `__BONUS` (as if it were a preprocessor macro).

Note: Rob says `__BONUS` = 2.