CSE 150: Problem Set #2

Problem 1

Find sets A and B satisfying each of the following condictions. Make your sets as small as possible.

1. $|A| < |B|, A \not\subset B$
2. $|P(A)| > 13$.
3. The function $f : A \to B,\ f(x) = \sin x$ is an injection.
4. $A \neq \emptyset$, and the relation $R \subseteq A \times B$ where $xRy \iff y = \sin x$ is an equivalence relation.

Problem 2

How would you change the definitions of the following relations to make them equivalence relations, and why?

1. $R \subseteq \text{People} \times \text{People}$. xRy iff x and y were born within 24 hours of each-other.
2. $R \subseteq \text{Cities} \times \text{Cities}$. xRy iff it is possible to fly directly from x to y.
3. $R \subseteq \text{People} \times \text{People}$. xRy iff x and y have the same mother and y is at least as old as x.

Problem 3

Describe and count the equivalence classes of each relation given below. Is each equivalence class the same size?

1. $R \subseteq \mathbb{Z} \times \mathbb{Z}$. xRy iff $x \bmod 7 = y \bmod 7$
2. $R \subseteq \mathbb{Z} \times \mathbb{Z}$. xRy iff $\lfloor \frac{x}{7} \rfloor = \lfloor \frac{y}{7} \rfloor$. (Definition: $\lfloor z \rfloor$ is the largest integer less than or equal to z)
3. Let $B = \{0, 1\}$, and B^n be the set of n-bit strings. $R \subseteq B^n \times B^n$. xRy iff x and y have the same number of 0s.

Problem 4

For each function, indicate whether it is injective, surjective, both, or neither.

1. $f : \mathbb{R}^+ \to \mathbb{R},\ f(x) = \log x$.
2. $f : \mathbb{N} \to \mathbb{R},\ f(x) = \cos \pi x$.
3. $f : \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\},\ f(x) = 3x \bmod 5$.
4. $f : \{0, 1, 2, 3, 4, 5\} \to \{0, 1, 2, 3, 4, 5\},\ f(x) = 3x \bmod 6$.

Problem 5

Write a formal proof that if $A \subseteq B$, then $|A| \leq |B|$, i.e. there exists a surjection $f : B \to A$.

Problem 6

Let $B = \{0, 1\}$. For $n \in \mathbb{N}$, we call B^n the binary strings of length n. Note the $B^1 = B$ and $B^0 = \{\epsilon\}$, where ϵ is the string of length 0. Let $B^* = \cup_{n=0}^{\infty} B^n$. Note that every string in B^* has finite length.

1. Prove, by induction on n, that $|B^n| = 2^n$.
2. Prove that B^* is countable.
3. Every computer program is a binary string. What is the size of the set of all computer programs?

Problem 7

Consider a simple programming language that only has three kinds of statements:

- `y:=E`, where `E` is some expression in terms of variables, +, -, $\times$, and /.
- `S; T`, which executes `S` and then `T`.
- `for (i = a ... b) S`, which executes `S` once for each integer in the range $[a, b]$. Note that a and b must be constants.

Prove that every program written in this language terminates (i.e. no program can run forever). In other words, prove that for every program P, there exists an integer N such that P always runs in N seconds or less.

Problem 8

- Prove that $\Sigma_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.
- How many breaks does it take to break an $m \times n$ piece of chocolate into 1×1 pieces? Prove your answer.

Bonus Problem

Prove that an integer is divisible by 9 iff the sum of its digits (in base 10) is divisible by 9.