

CSE 150: Problem Set #1

Due: Sep. 26, 2007

Problem 1

Write down the following sets, and their cardinalities (sizes):

Note: If you don't know the proper notation for something, just describe it as well as you can.

- $\{x \in \mathbb{Z} \mid |x| < 10\} \setminus \mathbb{N}$
 $|\{-9, -8, -7, \dots, -1\}| = 9$
- $\{\{6, 3, 24\}, \{\emptyset\}, \{\text{forty-blue}\}\} \triangle \{\emptyset, \{6\}\}$
 $|\{\{6, 3, 24\}, \{\emptyset\}, \{\text{forty-blue}\}, \emptyset, \{6\}\}| = 5$
- $P(\{5, \{6, 7, \{8\}\}, 9\})$
 $|\{\emptyset, \{5\}, \{\{6, 7, \{8\}\}\}, \{9\}, \{5, \{6, 7, \{8\}\}\}, \{5, 9\}, \{\{6, 7, \{8\}\}, 9\}, \{5, \{6, 7, \{8\}\}, 9\}\}| = 8$
- $\bigcap_{i=0}^{10} \{x \in \mathbb{N} \mid x \leq i\}$ (*Note: this is the “summation” notation for intersection*)
 $|\{0\}| = 1$
- $(S \times S) \times S$, where $S = \{3, 4\}$
 $|\{((3, 3), 3), ((3, 3), 4), ((3, 4), 3), ((3, 4), 4), ((4, 3), 3), ((4, 3), 4), ((4, 4), 3), ((4, 4), 4)\}| = 8$
- $P(P(P(\emptyset)))$
 $|\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}| = 4$
- $P(P(P(\emptyset) \setminus \{\emptyset\}) \setminus \emptyset) \setminus \{\emptyset\}$
 $|\{\{\emptyset\}\}| = 1$

Problem 2

Indicate whether each relation R is reflexive, symmetric, or transitive:

- $S = \{1, 2, 3, \dots, 10\}$,
 $R \subseteq S \times S$,
 $R = \text{“is less than the square of”}$
 - R is not reflexive because $(1, 1) \notin R$, since $1 \not< 1^2$.
 - R is not symmetric because $(1, 2) \in R$, since $1 < 2^2 = 4$, but $(2, 1) \notin R$, since $2 \not< 1^2$.
 - R is not transitive because $(8, 3) \in R$ ($8 < 3^2 = 9$), and $(3, 2) \in R$ ($3 < 2^2 = 4$), but $(8, 2) \notin R$ ($8 \not< 2^2 = 4$).
- $R \subseteq \mathbb{R} \times \mathbb{R}$,
 $(x, y) \in R \Leftrightarrow y = \sin x$
 - R is not reflexive because $10 \neq \sin 10$.

- (b) R is not symmetric because $\pi R 0$ but it is not the case that $0R\pi$.
- (c) R is not transitive because $\pi/2 R 1$ and $1 R \sin 1$, but not $\pi/2 R \sin 1$.
3. $R \subseteq \text{People} \times \text{People}$,
 $R = \text{"is the sibling of"}$

Note: we assume siblings must have both parents in common, and that everyone is their own sibling. Other interpretations are acceptable if they are clearly stated and reasonable.

- (a) Since everyone is their own sibling, R is reflexive.
- (b) R is symmetric. "Proof": If A is the sibling of B , then A and B have a mother and father in common. Hence B is also the sibling of A .
- (c) R is transitive. "Proof": If A and B are siblings, then A and B have mother and father, (M and F , respectively) in common. If B and C are siblings, then B and C have mother and father in common. Since everyone has exactly one mother and father, the mother and father B and C have in common must also be M and F . Thus A and C have M and F in common, too, and hence are siblings.

Problem 3

Recall that for a function f , $f(X) = \{y \mid y = f(x), x \in X\}$ and $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. For each of the following claims, either prove it is true or provide a counterexample. If it is not true of all functions, determine if it is true for surjections, injections, or bijections, and prove that it is.

1. $f(f^{-1}(Y)) = Y$

This is not true unless f is a surjection. Consider the function $f : \{0\} \rightarrow \{0, 1\}$, $f(x) = x$. Note that f is an injection, but not a surjection, and $f(f^{-1}(\{0, 1\})) = f(\{0\}) = \{0\} \neq \{0, 1\}$. Since the conjecture isn't true for injections, it must not be true for all functions, either.

Theorem 0.1 *Suppose $f : S \rightarrow T$ is a function and $Y \subseteq f(S)$. Then $f(f^{-1}(Y)) = Y$.*

Proof Let $y \in f(f^{-1}(Y))$. By definition, $y \in f(f^{-1}(Y))$ iff $\exists x \in f^{-1}(Y)$ s.t. $f(x) = y$. If $x \in f^{-1}(Y)$, then $f(x) = y \in Y$. Hence, $f(f^{-1}(Y)) \subseteq Y$. Now let $y \in Y$. Since $Y \subseteq f(S)$, $\exists x \in S$ s.t. $f(x) = y$. Such an x would be an element of $f^{-1}(Y)$, so $y \in f(f^{-1}(Y))$. Hence $Y \subseteq f(f^{-1}(Y))$. Combining these two facts, $Y = f(f^{-1}(Y))$. ■

Corollary 0.2 *If $f : S \rightarrow T$ is a surjection and $Y \subseteq T$, then $f(f^{-1}(Y)) = Y$.*

Proof Since f is a surjection, $Y \subseteq T = f(S)$, so by Theorem 0.1, $f(f^{-1}(Y)) = Y$. ■

Corollary 0.3 *If $f : S \rightarrow T$ is a bijection and $Y \subseteq T$, then $f(f^{-1}(Y)) = Y$.*

Proof Since f is a bijection, f is a surjection, so we can use Corollary 0.2. ■

2. $f^{-1}(f(X)) = X$

This is not true unless f is an injection. Consider $f : \{0, 1\} \rightarrow \{0\}$, $f(x) = 0$. Then f is a surjection but not an injection, and $f^{-1}(f(\{0\})) = f^{-1}(\{0\}) = \{0, 1\} \neq \{0\}$. Since the conjecture is not true of all surjections, it certainly cannot be true for all functions.

Definition A function $f : S \rightarrow T$ is *faithful* on a set $X \subseteq S$ iff for all $x \in S$, $f(x) \in f(X)$ implies that $x \in X$.

Theorem 0.4 *If $f : S \rightarrow T$ is faithful on a set X , then $f^{-1}(f(X)) = X$.*

Proof Let $x \in X$. Then $f(x) \in f(X)$, and hence $x \in f^{-1}(f(X))$. So $X \subseteq f^{-1}(f(X))$. Now let $x \in f^{-1}(f(X))$. Then $f(x) \in f(X)$ and, since f is faithful on X , $x \in X$. Thus $f^{-1}(f(X)) \subseteq X$. Together, these imply that $X = f^{-1}(f(X))$. ■

Corollary 0.5 *If $f : S \rightarrow T$ is injective and $X \subseteq S$, then $f^{-1}(f(X)) = X$.*

Proof If $f(x) \in f(X)$, then there exists an $x' \in X$ s.t. $f(x') = f(x)$. Since f is injective, $x = x' \in X$, so f is faithful on X . By theorem 0.4, $f^{-1}(f(X)) = X$. ■

Corollary 0.6 *If $f : S \rightarrow T$ is a bijection and $X \subseteq S$, then $f^{-1}(f(X)) = X$.*

Proof Since f is a bijection, f is an injection, so we can use Corollary 0.5. ■

3. $f(f^{-1}(f(X))) = f(X)$

Corollary 0.7 *If $f : S \rightarrow T$, then $f(f^{-1}(f(X))) = f(X)$.*

Proof It is always the case that $f(X) \subseteq f(S)$, so by Theorem 0.1, $f(f^{-1}(f(X))) = f(X)$. ■

4. $f^{-1}(f(f^{-1}(Y))) = f^{-1}(Y)$

Corollary 0.8 *If $f : S \rightarrow T$, then $f^{-1}(f(f^{-1}(Y))) = f^{-1}(Y)$.*

Proof Suppose $f(x) \in f(f^{-1}(Y))$. Then there exists $x' \in f^{-1}(Y)$ s.t. $f(x') = f(x)$. Thus $f(x) = f(x') \in Y$, so $x \in f^{-1}(Y)$. Hence f is faithful on $f^{-1}(Y)$. By Theorem 0.4, $f^{-1}(f(f^{-1}(Y))) = f^{-1}(Y)$. ■

5. $f(X \cup Y) = f(X) \cup f(Y)$

Theorem 0.9 *If $f : S \rightarrow T$, then for any sets $X, Y \subseteq S$, $f(X \cup Y) = f(X) \cup f(Y)$.*

Proof

$$\begin{aligned} z \in f(X \cup Y) &\iff \exists w \in X \cup Y, f(w) = z \\ &\iff \exists w \in X, f(w) = z \text{ or } \exists w \in Y, f(w) = z \\ &\iff z \in f(X) \text{ or } z \in f(Y) \\ &\iff z \in f(X) \cup f(Y) \end{aligned}$$

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6. $f(X \cap Y) = f(X) \cap f(Y)$

True only for injections. Let $f : \{0, 1\} \rightarrow \{0\}$ be the surjection $f(x) = 0$. Then $f(\{0\} \cap \{1\}) = f(\emptyset) = \emptyset$, but $f(\{0\}) \cap f(\{1\}) = \{0\} \cap \{0\} = \{0\}$.

Theorem 0.10 *If $f : S \rightarrow T$ is an injection, then for any sets $X, Y \subseteq S$, $f(X \cap Y) = f(X) \cap f(Y)$.*

Proof If $z \in f(X \cap Y)$, then there exists $x \in X \cap Y$ such that $f(x) = z$. Since $x \in X \cap Y$, $x \in X$ and $x \in Y$. Hence, $z = f(x) \in f(X)$ and $z \in f(Y)$, so $z \in f(X) \cap f(Y)$. In conclusion, $f(X \cap Y) \subseteq f(X) \cap f(Y)$. Now let $z \in f(X) \cap f(Y)$. Then there exists $x_1 \in X$ s.t. $f(x_1) = z$ and there exists $x_2 \in Y$ s.t. $f(x_2) = z$. Since f is injective, $x_1 = x_2 \in X \cap Y$. Hence $z \in f(X \cap Y)$, so $f(X) \cap f(Y) \subseteq f(X \cap Y)$. Putting these together, $f(X) \cap f(Y) = f(X \cap Y)$. ■

Corollary 0.11 *If $f : S \rightarrow T$ is a bijection, then for any sets $X, Y \subseteq S$, $f(X \cap Y) = f(X) \cap f(Y)$.*

Proof Every bijection is an injection, so Theorem 0.10 applies. ■

7. $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$

Theorem 0.12 *If $f : S \rightarrow T$ and $X, Y \subseteq T$, then $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$.*

Proof

$$\begin{aligned} z \in f^{-1}(X \cup Y) &\iff f(z) \in X \cup Y \\ &\iff f(z) \in X \text{ or } f(z) \in Y \\ &\iff z \in f^{-1}(X) \text{ or } z \in f^{-1}(Y) \\ &\iff z \in f^{-1}(X) \cup f^{-1}(Y) \end{aligned}$$

■

8. $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

Theorem 0.13 *If $f : S \rightarrow T$ and $X, Y \subseteq T$, then $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.*

Proof

$$\begin{aligned} z \in f^{-1}(X \cap Y) &\iff f(z) \in X \cap Y \\ &\iff f(z) \in X \text{ and } f(z) \in Y \\ &\iff z \in f^{-1}(X) \text{ and } z \in f^{-1}(Y) \\ &\iff z \in f^{-1}(X) \cap f^{-1}(Y) \end{aligned}$$

■

Problem 4

Let $n \geq 2$ be an integer, and define a relation on the integers by $a \equiv b$ if n divides $a - b$. Prove that \equiv is an equivalence relation. (Def: n divides x iff there exists an integer k such that $x = kn$)

Theorem 0.14 \equiv is an equivalence relation.

Proof To show that \equiv is an equivalence relation, we must show that it is reflexive, symmetric and transitive.

- (Reflexivity) For any $a \in \mathbb{Z}$, $a - a = 0 = 0 \times n$, so $n|a - a$, and hence $a \equiv a$.
- (Symmetry) Let $a, b \in \mathbb{Z}$ s.t. $a \equiv b$. Then $n|a - b$, and hence there is a $k \in \mathbb{Z}$ s.t. $kn = a - b$. Consequently, $(-k) \times n = b - a$, so $n|b - a$, too. Thus $b \equiv a$.
- (Transitivity) Let $a, b, c \in \mathbb{Z}$ s.t. $a \equiv b$ and $b \equiv c$. Then there exist $k_1, k_2 \in \mathbb{Z}$ s.t. $k_1n = a - b$ and $k_2n = b - c$. Therefore, $(k_1 + k_2)n = a - c$, so $a \equiv c$.