CSE 150: Problem Set #2

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Problem 1

Find sets A and B satisfying each of the following condictions. Make your sets as small as possible.

1. $|A| < |B|, A \not\subset B$
2. $|P(A)| > 13$.
3. The function $f : A \to B$, $f(x) = \sin x$ is an injection.
4. $A \neq \emptyset$, and the relation $R \subseteq A \times B$ where $xRy \iff y = \sin x$ is an equivalence relation.

Possible Solutions

1. $A = \{\varnothing\}, B = \{\{\varnothing\}, \{\{\varnothing\}\}\}$
2. $A = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}\}$
3. $A = B = \{0\}$ (or $A = B = \varnothing$)
4. $A = B = \{0\}$

Problem 2

How would you change the definitions of the following relations to make them equivalence relations, and why?

1. $R \subseteq$ People $\times$ People. xRy iff x and y were born within 24 hours of each-other.
2. $R \subseteq$ Cities $\times$ Cities. xRy iff it is possible to fly directly from x to y.
3. $R \subseteq$ People $\times$ People. xRy iff x and y have the same mother and y is at least as old as x.

Possible Solutions

1. xRy iff x and y were born on the same day (00:00:00 through 23:59:59) by the UTC/GMT timezone, because it wasn't transitive before, and so didn't "partition" the set like it should have.
2. xRy iff it is possible to fly directly or indirectly from x to y.
3. xRy iff x and y have the same mother and x and y are the same age, because it wasn't symmetric before.

Problem 3

Describe and count the equivalence classes of each relation given below. Is each equivalence class the same size?

1. $R \subseteq \mathbb{Z} \times \mathbb{Z}$. xRy iff $x \bmod 7 = y \bmod 7$
2. $R \subseteq \mathbb{Z} \times \mathbb{Z}$. xRy iff $\lfloor \frac{x}{7} \rfloor = \lfloor \frac{y}{7} \rfloor$. (Definition: $\lfloor z \rfloor$ is the largest integer less than or equal to z)
3. Let $B = \{0, 1\}$, and B^n be the set of n-bit strings. $R \subseteq B^n \times B^n$. xRy iff x and y have the same number of 0s.

Possible Solutions

1. The equivalence classes are $[0]_7 = \{7k | k \in \mathbb{Z}\}, [1]_7 = \{7k + 1 | k \in \mathbb{Z}\}, [2]_7, \ldots, [6]_7$, there are 7 of them, and they are all the same size ($\aleph_0$)
2. The equivalence classes are all sets $\{7n + i | 0 \leq i < 7 \in \mathbb{N}\}$, where n is any integer. There are a countably infinite ($\aleph_0$) number of them, and they all have 7 elements.
3. There are $n + 1$ equivalence classes, such that there is an equivalence class of strings with i 0s for each $0 \leq i \leq n \in \mathbb{N}$. They are not all the same size. For example, there is 1 string with no 0s in B^2 but 2 strings with 1 0. (In general, each equivalence class has $\binom{n}{i} = \frac{n!}{(n-i)!}$ elements, but you did not need to prove this in your homework.)

Problem 4

For each function, indicate whether it is injective, surjective, both, or neither.

1. $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log x$.
   (Note the correction from the original version.)
   injective, surjective, bijective.
2. $f : \mathbb{N} \to \mathbb{R}$, $f(x) = \cos \pi x$.
   none
3. $f : \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$, $f(x) = 3x \bmod 5$.
   injective, surjective, bijective
4. $f : \{0, 1, 2, 3, 4, 5\} \to \{0, 1, 2, 3, 4, 5\}$, $f(x) = 3x \bmod 6$.
   none

Problem 5

Write a formal proof that if $A \subseteq B$, then $|A| \leq |B|$, i.e. there exists a surjection $f : B \to A$.

Proof If $A = \emptyset$, then $|A| = 0$, so $|A| \leq |B|$. If $A \neq \emptyset$, let $a \in A$. Define $f : B \to A$ by $f(x) = x$ if $x \in A$ and $f(x) = a$ otherwise. Since $A \subset B$, for all $z \in A$, $z = f(z)$, so f surjects onto A. Hence $|A| \leq |B|$.

Problem 6

Let $B = \{0, 1\}$. For $n \in \mathbb{N}$, we call B^n the binary strings of length n. Note the $B^1 = B$ and $B^0 = \{\epsilon\}$, where ϵ is the string of length 0. Let $B^* = \cup_{n=0}^{\infty} B^n$. Note that every string in B^* has finite length.

1. Prove, by induction on n, that $|B^n| = 2^n$.

   Proof We induct on n.

- **Base case.** For $n = 0$, $B^n = B^0 = \{\epsilon\}$, so $|B^0| = 1$.
- **Inductive case.** Suppse $|B^n| = 2^n$. Then $|B^{n+1}| = |B^n \times B| = |B^n| \times |B| = 2^n \times 2 = 2^{n+1}$.

2. Prove that B^* is countable.

   Proof Let 0^n be the string consisting of n 0s. Note that B^* is infinite because it contains 0^i for all $i \in \mathbb{N}$.

   On the other hand, observe that $B^i \subset \mathbb{N}^i$, and $\mathbb{N}^i$ is countable from a theorem in class. Thus $\mathbb{N}^* = \cup_{n=0}^{\infty} \mathbb{N}^i$ is countable, since we proved in class that a countable union of countable sets is countable. Since $B^* \subset \mathbb{N}^*$, by Problem 5 above, $|B^*| \leq |\mathbb{N}^*|$. So B^* is at most countable.

   Since B^* is infinite but at most countable, it must be countable.

3. Every computer program is a binary string. What is the size of the set of all computer programs? Countable. This means that, for example, there exist real numbers which cannot be computed by any program.

Problem 7

Consider a simple programming language that only has three kinds of statements:

- `y:=E`, where `E` is some expression in terms of variables, +, -, ×, and /.
- `S; T`, which executes `S` and then `T`.
- `for (i = a ... b) S`, which executes `S` once for each integer in the range $[a, b]$.

Prove that every program written in this language terminates (i.e. no program can run forever). In other words, prove that for every program P, there exists an integer N such that P always runs in N seconds or less.

Proof We proceed by structural induction.

1. Statements of the form `y:=E` complete in a finite amount of time since `E` only contains a finite number of operations, each of which takes a finite amount of time.
2. Now suppose `P` is of the form `S;T`. By induction, there exist integers M and N such that `S` and `T` always complete within S and T steps, respectively. Thus `P` will complete in $M + N$ steps.
3. Suppose `P` is of the form `for(i=a...b) S`. By induction, there exists an integer N such that `S` always completes within N steps. Consequently, `P` will complete within $(b - a + 1)N$ steps.

Problem 8

- Prove that $\Sigma_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

  Proof We induct on n

  - **Base case.** For $n = 0$, $\Sigma_{i=1}^0 i^3 = 0 = \left[\frac{0(0+1)}{2}\right]^2$
  - **Inductive step.** Assume $\Sigma_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$. Then

$$\begin{aligned}\Sigma_{i=1}^{n+1} i^3 &= \Sigma_{i=1}^n i^3 + (n+1)^3 \\ &\overset{(IHOP)}{=} \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \\ &= \frac{n^4+2n^3+n^2}{4} + n^3 + 3n^2 + 3n + 1 \\ &= \frac{n^4+6n^3+13n^2+12n+4}{4}\end{aligned}$$

  On the other hand, $\left[\frac{(n+1)(n+2)}{2}\right]^2 = \frac{n^4+6n^3+13n^2+12n+4}{4}$, so $\Sigma_{i=1}^{n+1} i^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2$.

- How many breaks does it take to break an $m \times n$ piece of chocolate into 1×1 pieces? Prove your answer.

  Proof Every break increases the number of pieces by 1. Since we start with 1 piece and end with mn, we must perform $mn - 1$ breaks.

Bonus Problem

Prove that an integer is divisible by 9 iff the sum of its digits (in base 10) is divisible by 9.

Proof For this proof, we will use two facts alluded to in class: $a + b \bmod 9 = (a \bmod 9 + b \bmod 9) \bmod 9$, and $a \times b \bmod 9 = (a \bmod 9 \times b \bmod 9) \bmod 9$. For an integer $z = d_n d_{n-1} \cdots d_0$,

$$
\begin{aligned}
z \bmod 9 &= (\Sigma_{i=0}^{n} d_i 10^i) \bmod 9 \\
&= (\Sigma_{i=0}^{n} d_i 10^i \bmod 9) \bmod 9 \\
&= (\Sigma_{i=0}^{n} (d_i \bmod 9)((10 \bmod 9)^i) \bmod 9) \bmod 9 \\
&= (\Sigma_{i=0}^{n} (d_i \bmod 9)(1^i \bmod 9)) \bmod 9 \\
&= (\Sigma_{i=0}^{n} (d_i \bmod 9)(1 \bmod 9)) \bmod 9 \\
&= (\Sigma_{i=0}^{n} (d_i \bmod 9)) \bmod 9 \\
&= (\Sigma_{i=0}^{n} d_i) \bmod 9
\end{aligned}
$$

Also, z is divisable by 9 iff $z \bmod 9 = 0$. Thus $9|z$ iff $9|\Sigma_{i=0}^{n} d_i$