

# CSE 150 Fall 2009: Homework #3

*Note: If you don't know the proper notation for something, just describe it as well as you can.*

Hint: Use induction.

## Problem 1

Recall that for a function  $f : A \rightarrow B$  and  $X \subseteq A$  and  $Y \subseteq B$ ,

$$f(X) = \{y \in B \mid y = f(x), x \in X\}$$

and

$$f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}$$

For each pair of sets listed below, indicated whether the first is always a subset of ( $\subseteq$ ), strict subset of ( $\subset$ ), superset of ( $\supseteq$ ), strict superset of ( $\supset$ ), equal to ( $=$ ), or not equal to ( $\neq$ ) the second subset, for each type of function. Make the strongest claim that you can.

Set 1	Set 2	Any function	Injection	Surjection	Bijection
$f(f^{-1}(Y))$	$Y$				
$f^{-1}(f(X))$	$X$				
$f(f^{-1}(f(X)))$	$f(X)$				
$f^{-1}(f(f^{-1}(Y)))$	$f^{-1}(Y)$				
$f(X_1 \cup X_2)$	$f(X_1) \cup f(X_2)$				
$f(X_1 \cap X_2)$	$f(X_1) \cap f(X_2)$				
$f^{-1}(Y_1 \cup Y_2)$	$f^{-1}(Y_1) \cup f^{-1}(Y_2)$				
$f^{-1}(Y_1 \cap Y_2)$	$f^{-1}(Y_1) \cap f^{-1}(Y_2)$				

## Problem 2

An Eulerian cycle is a path through a graph that crosses each edge exactly once. Suppose  $G$  is a connected graph. Prove that  $G$  has an Eulerian cycle if and only if every vertex of  $G$  has even degree.

## Problem 3

Let  $k$  be a fixed positive integer. Put the following functions in increasing asymptotic order. Indicate which functions, if any, are asymptotically equivalent (i.e.  $\Theta()$ ).

$$\begin{aligned}
 f_1(n) &= (\log n)^{\log n} \\
 f_2(n) &= 5\sqrt{n} \\
 f_3(n) &= \sum_{i=0}^k n^i \\
 f_4(n) &= n^{k+1} \\
 f_5(n) &= (\log n)^{\log \log n} \\
 f_6(n) &= 2^n \\
 f_7(n) &= \sum_{i=0}^n i^k \\
 f_8(n) &= \binom{n}{k}
 \end{aligned}$$

## Problem 4

As you have noticed, a proof of a theorem consists of a finite sequence of letters and symbols. Let  $S$  be the set of all symbols that can occur in a proof, e.g.  $S$  would contain all the English letters and punctuation, Greek letters, and various mathematical symbols. Argue that if some theorem has a proof, you can always find the proof eventually.

## Problem 5

If  $A[0, \dots, n-1]$  is a permutation of  $0, \dots, n-1$ , then we can view  $A$  as a function  $A : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$ . Write an algorithm that finds  $B[0, \dots, n-1]$  such that  $A \circ B$  is the identity function.

## Problem 6

Use the partition algorithm to write a function **find-kth-smallest**( $A, n, k$ ) that, given an array,  $A$ , of  $n$  integers, returns the  $k$ th smallest integer in the array. In other words, your algorithm should return the integer that would be in position  $k$  after the array is sorted. A trivial solution to this problem is

```
procedure find-kth-smallest( $A, n, k$ )
    qsort( $A, n$ )
    return  $A[k]$ 
```

This algorithm would have running time  $O(n \log n)$ . Your algorithm should be faster.