Computer Science 547

Midterm Take-Home Examination 1
Due Thursday, February 25, 1999

February 18, 1999

This exam is open book but closed mouth, meaning that you may use the textbook but not discuss it with anyone but Steve Skiena or Xie Hui. As far as we know, the answers to these problems do not exist in any book, so it is both a waste of time and violation of the spirit of the exam to prowl through books looking for answers.

Please write out and sign the University of Virginia Honor Pledge on your exam: "On my honor as a student, I have neither given nor received aid on this examination".

Each of the six problems should be solved on a separate sheet of paper to facilitate grading. Staple all these sheets together! Show your work - magic guesses to a solution may not receive full credit. Don't wait until the last minute. Good luck.

1. What is the last digit of 1777^{1777}? You must show your work.

2. Prove that Euclid's algorithm takes at most $2\log_2 n + 1$ iterations to find $GCD(m, n)$, where $n < m$.

3. Consider the set of strings of length n on the alphabet $\{1, 2, \ldots, k\}$. A string is *one-even* if it contains an even number of 1's, and let $M(n, k)$ count the number of one-even strings of length n on the alphabet. For example, $M(2, 4) = 10$.

   (a) Give a recurrence to compute $M(n, k)$.

   (b) Solve this recurrence.

4. A balance scale computes the difference between the weight on each side, and tilts towards the heavier side unless they are equal. Using a predefined set of integer weights, we can determine a variety of different integers by placing various subsets on either side. For example, the set of weights $\{2, 5, 6\}$ can measure any integer weight between 1 and 8 (note that 1 is realized by putting 5 and 6 on different sides, while 7 is realized by summing 2 and 5). In general, how many weights $w(n)$ are necessary to weigh, by using a balance, any object of integer weight between 1 and n? Give a construction for their weights which realizes this bound. Partial credit will be given for upper and lower bounds.

5. Solve

$$\sum_{1 \leq j \leq n!} \sum_{1 \leq k \leq n} \left(\left\lfloor \frac{j \cdot k!}{n!} \right\rfloor \bmod k\right).$$

It might help to think about what the sum is doing instead of trying to solve it mechanically.

6. Consider the first n terms of a Beatty sequence $\lfloor \alpha i \rfloor$, where $1 \leq i \leq n$ and $0 \leq \alpha \leq 1$. Since n is finite, different rational numbers may generate the same sequence.

   (a) A Farey fraction of order n is a reduced simple fraction whose denominator is at most n. Prove that if α and β are distinct Farey fractions of order n, then $\lfloor \alpha i \rfloor \neq \lfloor \beta i \rfloor$ for some $1 \leq i \leq n$.

   (b) Prove that for any number α, the first n terms of its Beatty sequence are identical to the first n terms of the Beatty sequence of some Farey fraction of order n.

   (Hint: Equation 4.31 in the text may be useful for this problem.)