Computer Science 547

Midterm Take-Home Examination 2
Due Tuesday, April 6, 1999

March 25, 1999

This exam is open book but closed mouth, meaning that you may use the textbook but not discuss it with anyone but Steve Skiena or Xie Hui. As far as we know, the answers to these problems do not exist in any book, so it is both a waste of time and violation of the spirit of the exam to prowl through books looking for answers.

Please write out and sign the University of Virginia Honor Pledge on your exam: "On my honor as a student, I have neither given nor received aid on this examination".

Each of the six problems should be solved on a separate sheet of paper to facilitate grading. Staple all these sheets together! Show your work - magic guesses to a solution may not receive full credit. Don't wait until the last minute. Good luck.

1. Let $f(n)$ be the number of ways to tile a $3 \times n$ region with 1×3 dominos. Use generating functions to determine $f(n)$.

2. Evaluate the following sum S_n:

$$S_n = \sum_{k=0}^{n} \frac{\binom{n+k}{n}}{2^k}$$

3. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$, $T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}$. The first few terms are $\{2, 3, 6, 14, 40, 152, 784\}$. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences. (Hint: this sequence is in Sloane's Handbook on the WWW)

4. Solve the recurrence $x_{n+1} = 1 + \lfloor {x_n}^2 / x_{n-1} \rfloor$ where $x_0 = 1$ and $x_1 = 2$. You must prove your answer is correct. (Hint: think Fibonacci)

5. Prove that $\sum_{i=k}^{n} i \binom{i}{k} = n \binom{n+1}{k+1} - \binom{n+1}{k+2}$

6. The doorway to the CS Faculty offices is guarded by a five button lock. It can be considered a special instance of a general k button lock, where each number occurs at most once, and up to k buttons can be pressed at once. For a two button lock, the legal combinations are 1, 2, 12, 1-2, or 2-1. There are 25 legal combinations for a three-button lock.

   (a) List the 25 different combinations for a three-button lock.

   (b) Give an expression for computing $C(k)$, the number of combinations for a k button lock. Evaluate this for $C(5)$ and, if possible, for $C(12)$.