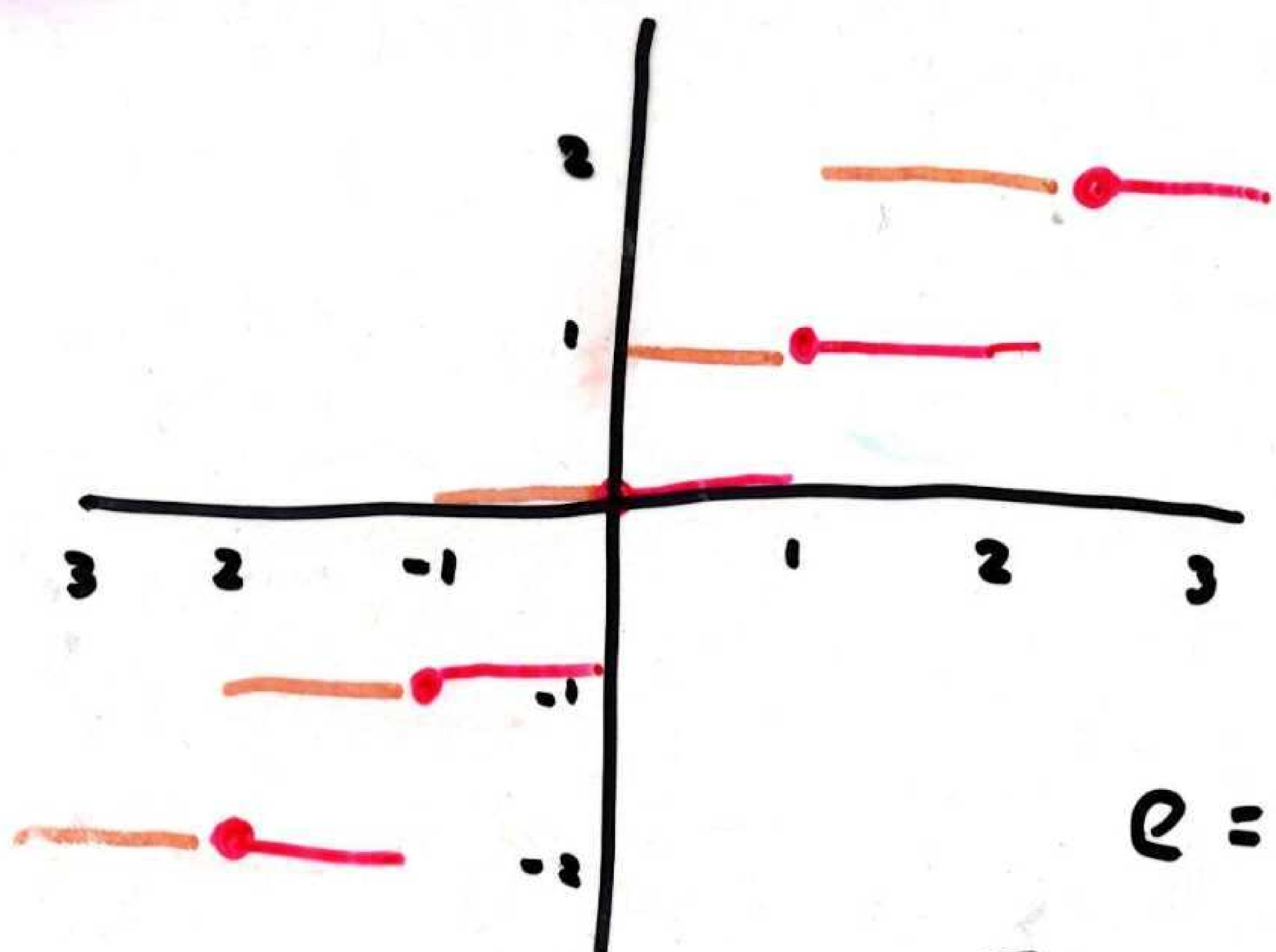
Floors and Ceilings

"Measure with a micrometer,
Mark with chalk,
Cut with an axe."

A mathematical axe for dealing with real numbers is the floor and ceiling functions.

$\lfloor x \rfloor$ = greatest integer less than or equal to x.

$\lceil x \rceil$ = least integer greater than or equal to x.



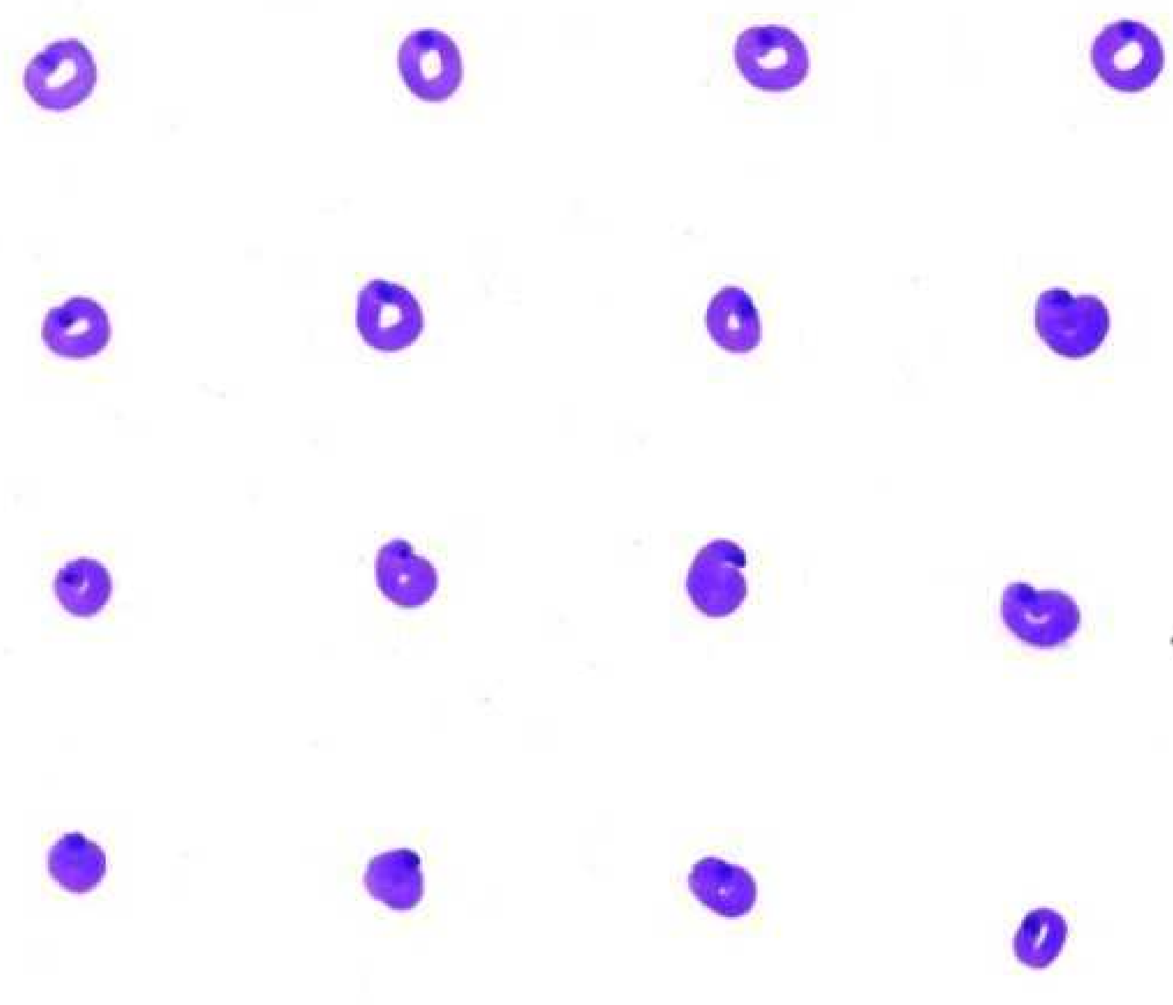
$e = 2.718\ldots$

$\lceil e \rceil = 3$

$\lfloor e \rfloor = 2$

$$\left.\begin{array}{l} \lceil -e \rceil = -2 \\ \lfloor -e \rfloor = -3 \end{array}\right\} \text{check these carefully}$$

Frequencies of Large Distances in Integer Lattices



How often does the i^{th} largest distance occur in an $N \times N$ integer lattice?

I had observed that this frequency is independent of N for the $N-1$ largest distances:

2, 8, 12, 8, 16, 24, 20, 32, 18, 24, 40, 48 ....

Venugopal Reddy, in his graduate project for this class proved the frequency of the i^{th} largest distance, $i < N$ is:

$2i$ if i is perfect square

$$4\left(i - \left\lfloor \frac{\lfloor 2\sqrt{i} \rfloor^2}{4} \right\rfloor\right)\left(\lceil 2\sqrt{i} \rceil - i + \left\lfloor \frac{\lfloor 2\sqrt{i} \rfloor^2}{4} \right\rfloor\right)$$

otherwise

Deriving such functions requires careful manipulation of floor + ceiling functions.

Properties of Floors and Ceilings

$\lfloor x \rfloor = \lceil x \rceil$ only when x is an integer, otherwise they differ by one

$\lceil x \rceil - \lfloor x \rfloor = (x \text{ is not an integer})$

in Iverson's notation

$$\underbrace{x-1 < \lfloor x \rfloor}_{} \le x \le \underbrace{\lceil x \rceil < x+1}_{}$$

Note strict inequality: to convince yourself, try integer + real values of x

$\lfloor -x \rfloor = -\lceil x \rceil$; $\lceil -x \rceil = -\lfloor x \rfloor$

floors and ceilings can be defined in terms of each other.

$\lfloor x + n \rfloor = \lfloor x \rfloor + n$, integer n

The fractional part of x is denoted $\{x\}$

$\{x\} = x - \lfloor x \rfloor$. this gives us a notation to reason rigorously about floors.

Although some sums become easier with $\lfloor\rfloor$, $\lceil\rceil$...

$$\sum_{i=1}^{\infty} \left\lfloor \frac{1}{i^3+49i^2} \right\rfloor = 0$$

$$\sum_{i=1}^{\infty} \left\lceil \frac{1}{i^3+49i^2} \right\rceil = \infty$$

most of the time they are trouble to work with because they cannot be factored:

$$\lfloor Nx \rfloor \neq N\lfloor x \rfloor$$

To do anything with them, we must treat floors and ceilings as inequalities and manipulate them using the Iversonian notation:

$$\lfloor x \rfloor = N \iff N \leq x < N+1$$

$$\lceil x \rceil = N \iff N-1 < x \leq N$$

$$\lfloor x \rfloor = N \iff x-1 < N \leq x$$

$$\lceil x \rceil = N \iff x \leq N < x+1$$

Carefully convince yourself of these!

Under certain circumstances floors or ceilings are redundant:

$$\lceil \lfloor x \rfloor \rceil \iff \lfloor x \rfloor$$
$$\lfloor \lceil x \rceil \rfloor \iff \lceil x \rceil$$

$$\left.\begin{array}{lll} x < N \iff \lfloor x \rfloor < N & & (a) \\ N < x \iff N < \lceil x \rceil & & (b) \\ x \le N \iff \lceil x \rceil \le N & & (c) \end{array}\right\} \text{integer } N, \text{ real } x.$$
$$N \le x \iff N \le \lfloor x \rfloor \qquad (d)$$

Manipulating floors and ceilings is a subtle game of trading off $\lfloor\rfloor$, $\lceil\rceil$ for relaxing the associated inequalities.

Once you have gotten as far as you can, conside doing a case analysis on the junk left in $\lfloor\rfloor$. Is it odd, even, etc?

Prove $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$, real $x \geq 0$:

Let $M = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$, note M is an integer

this implies $\quad M \leq \sqrt{\lfloor x \rfloor} < M+1$

Since $M \geq 0$, $\quad M^2 \leq \lfloor x \rfloor < (M+1)^2$

the floor can be dropped without changing anything! (d) (a)

$$M^2 \leq x < (M+1)^2$$

Now we can take the $\sqrt{}$ of everything:

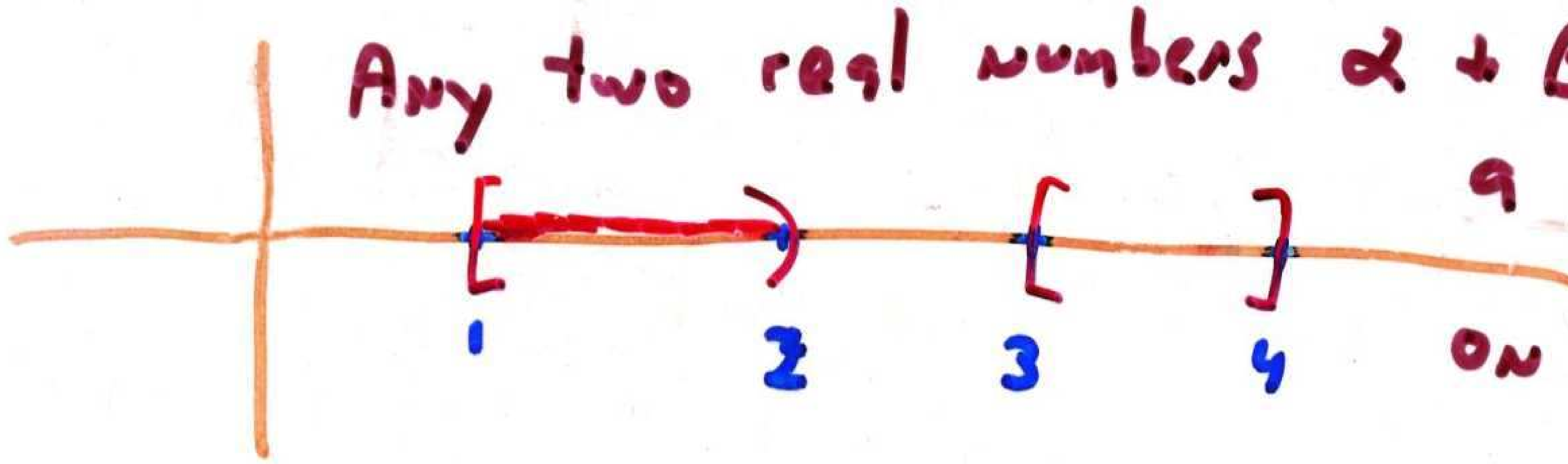
$$M \leq \sqrt{x} < (M+1)$$

By the definition of the floor function:

$$M = \lfloor \sqrt{x} \rfloor$$

$$\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$$

Intervals on a Line



Any two real numbers α + β specify a real interval on the real line

Intervals are open if they do not contain their endpoints, closed if they do, and half-open if they contain one of them.

$(\alpha, \beta]$ describes all x, $\alpha < x \leq \beta$

open   closed

$$(\alpha, \beta] \cup (\beta, \gamma] = (\alpha, \gamma]$$

half-open intervals are additive, open or closed are not.

Fairly often, we will need to know how many integers lie in an interval, and the question of open or closed can make a difference of up to 2.

If α + β are distinct integers, then

$[\alpha, \beta]$ contains $\beta - \alpha + 1$

(α, β) contains $\beta - \alpha - 1$

$[\alpha, \beta)$, $(\alpha, \beta]$ contains $\beta - \alpha$

when they are not integers:

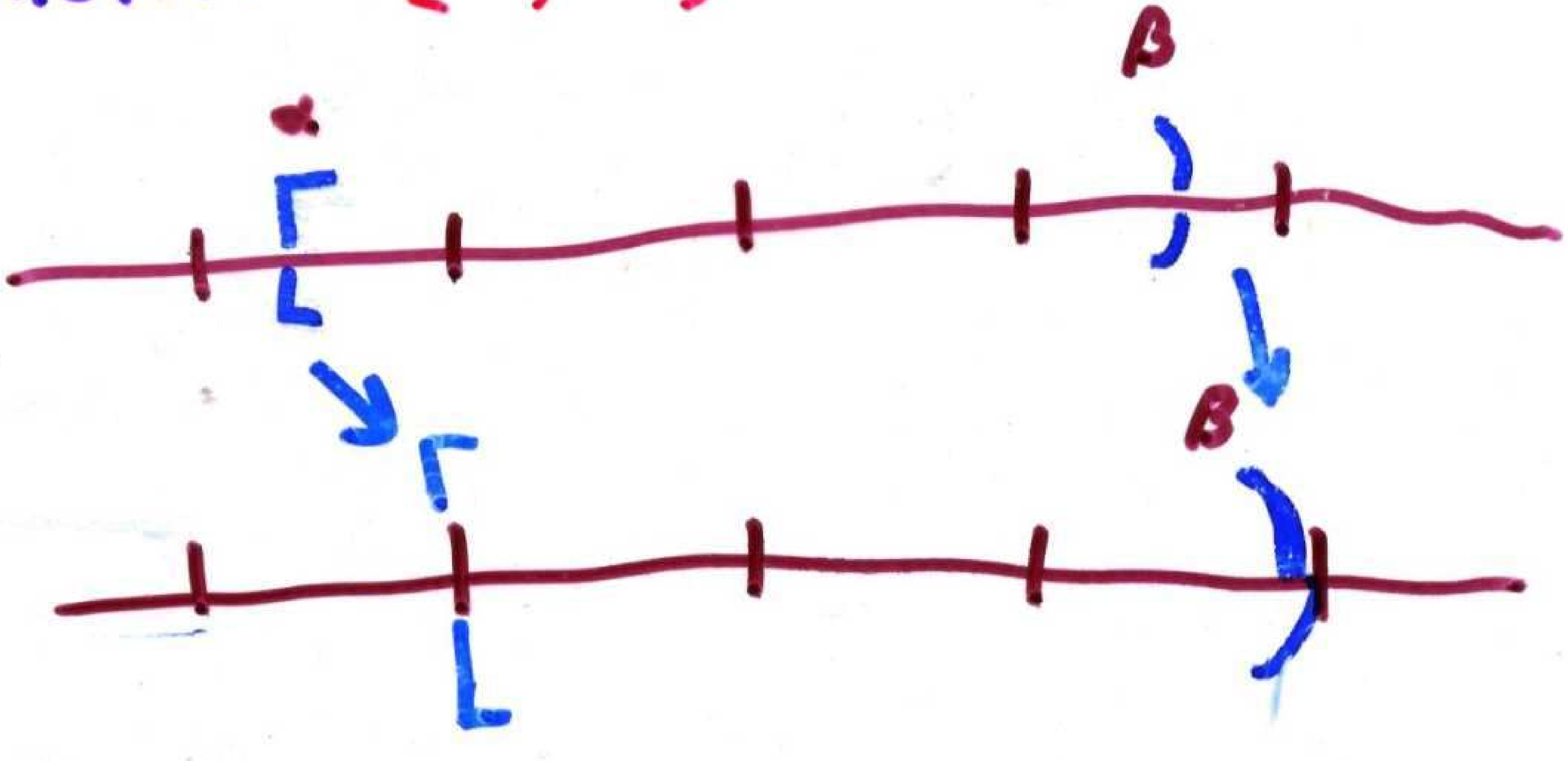
$[\alpha, \beta]$, $\alpha \le \beta$ contains $\lfloor \beta \rfloor - \lceil \alpha \rceil + 1$

$[\alpha, \beta)$ $\alpha \le \beta$ contains $\lceil \beta \rceil - \lceil \alpha \rceil$

$(\alpha, \beta]$ $\alpha \le \beta$ contains $\lfloor \beta \rfloor - \lfloor \alpha \rfloor$

(α, β) $\alpha < \beta$ contains $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$

The proofs that convince me easiest are geometric: $[\alpha, \beta)$



How many integers $1 \le N \le 1000$ satisfy $\lfloor \sqrt[3]{N} \rfloor \backslash N$? ie. the $\lfloor \sqrt[3]{N} \rfloor$ divides N.

$$W = \sum_{1 \le N \le 1000} \left(\lfloor \sqrt[3]{N} \rfloor \backslash N \right)$$

substitute $k = \lfloor \sqrt[3]{N} \rfloor$.

$$= \sum_{k,N} \left(k = \lfloor \sqrt[3]{N} \rfloor \right) (k \backslash N)(1 \le N \le 1000)$$

eliminate floor and divides

$$= \sum_{k,m,N} \left(k^3 \le N < (k+1)^3 \right)(N = km)(1 \le N \le 1000)$$

eliminate N, treat $k=10$ as a special case

$$= 1 + \sum_{k,m} \left(k^3 \le km < (k+1)^3 \right)(1 \le k < 10)$$

replace division by membership in an interval

$$= 1 + \sum_{k,m} \left(m \in [k^2, (k+1)^3/k) \right)(1 \le k < 10)$$

count members of the half-open interval

$$= 1 + \sum_{1 \le k < 10} \left(\lceil k^2 + 3k + 3 + 1/k \rceil - \lceil k^2 \rceil \right)$$

drop ceilings on all but $1/k$

$$= 1 + \sum_{1 \le k < 10} 3k + 4 = 172$$

Beatty Sequences

The Beatty sequence or spectra of a real number α is the sequence $\lfloor \alpha i \rfloor$, $i > 0$.

For a given number α, this sequence will be roughly linear in α, but with a skip when $\{a\}i$

$\alpha = \pi$ $\{3, 6, 9, 12, 15, 18, 21, 25, 28, 31 \ldots$

$\alpha = \pi/\pi-1$ $\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, \ldots$
$1.47\ldots$

$\alpha = e \approx 2.718$ $\{2, 5, 8, 10, 13, 16, 19, 21, 24, 27 \ldots)$

$\alpha = e/e-1$ $\{1, 3, 4, 6, 7, 9, 11, 12, 14, 15, 17, 18, 20 \ldots)$

$\alpha = \sqrt{2}$ $\{1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19 \ldots$

$\alpha = \sqrt{2}/\sqrt{2}-1$ $\{3, 6, 10, 13, 17, 20, 23, 27, \ldots$

For all rational α, the skip pattern will eventually repeat, between whenever $\alpha = \frac{M}{N}$ $kN \leq i \leq (k+1)N$.

But irrational α will not generate a repeating pattern.

Amazingly, the sets of sequences given above <u>partition</u> the positive integers exactly:

1. No integer appears in both sequences
2. each integer appears in one sequence.

Even more amazingly, this holds for ANY pair of irrational numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1 \quad !$$

Note $\alpha, \beta > 1$
Rational α, β will share sequence elements

To prove this, we can count the members of each sequence $\leq N$:

$$N(\alpha, N) = \sum_{k>0} (\lfloor k\alpha \rfloor \leq N)$$

$$= \sum_{k>0} (\lfloor k\alpha \rfloor < N+1)$$ since $N+1$ is an integer

$$= \sum_{k>0} (k\alpha < N+1)$$

$$= \sum_{k} \left(0 < k < \frac{N+1}{\alpha}\right)$$ Number of integers k in the interval

$$= \lceil (N+1)/\alpha \rceil - 1$$

To prove the two sequences form a partition, all we need do is show

$$\left\lceil \frac{(N+1)}{\alpha} \right\rceil - 1 + \left\lceil \frac{(N+1)(\alpha-1)}{\alpha} \right\rceil - 1 = N$$

since this shows that exactly one element is added to exactly one sequence as $N \to N+1$

$$\left\lceil \frac{(N+1)}{\alpha} \right\rceil - 1 + \left\lceil \frac{(N+1)(\alpha-1)}{\alpha} \right\rceil - 1$$

$$= \left\lfloor \frac{N+1}{\alpha} \right\rfloor + \left\lfloor \frac{(N+1)(\alpha-1)}{\alpha} \right\rfloor$$

Because α is irrational, nothing under the ceilings is ever an integer!

$$= \frac{N+1}{\alpha} - \left\{ \frac{N+1}{\alpha} \right\} + \frac{(N+1)(\alpha-1)}{\alpha} - \left\{ \frac{(N+1)(\alpha-1)}{\alpha} \right\}$$

$$= N+1 - \underbrace{\left[\left\{ \frac{N+1}{\alpha} \right\} + \left\{ \frac{(N+1)(\alpha-1)}{\alpha} \right\} \right]}$$

Thus the result holds if these fractional parts sum to 1.

This is a special case of

$$\left\{\frac{x}{\alpha}\right\} + \left\{\frac{x(\alpha-1)}{\alpha}\right\} = 1$$

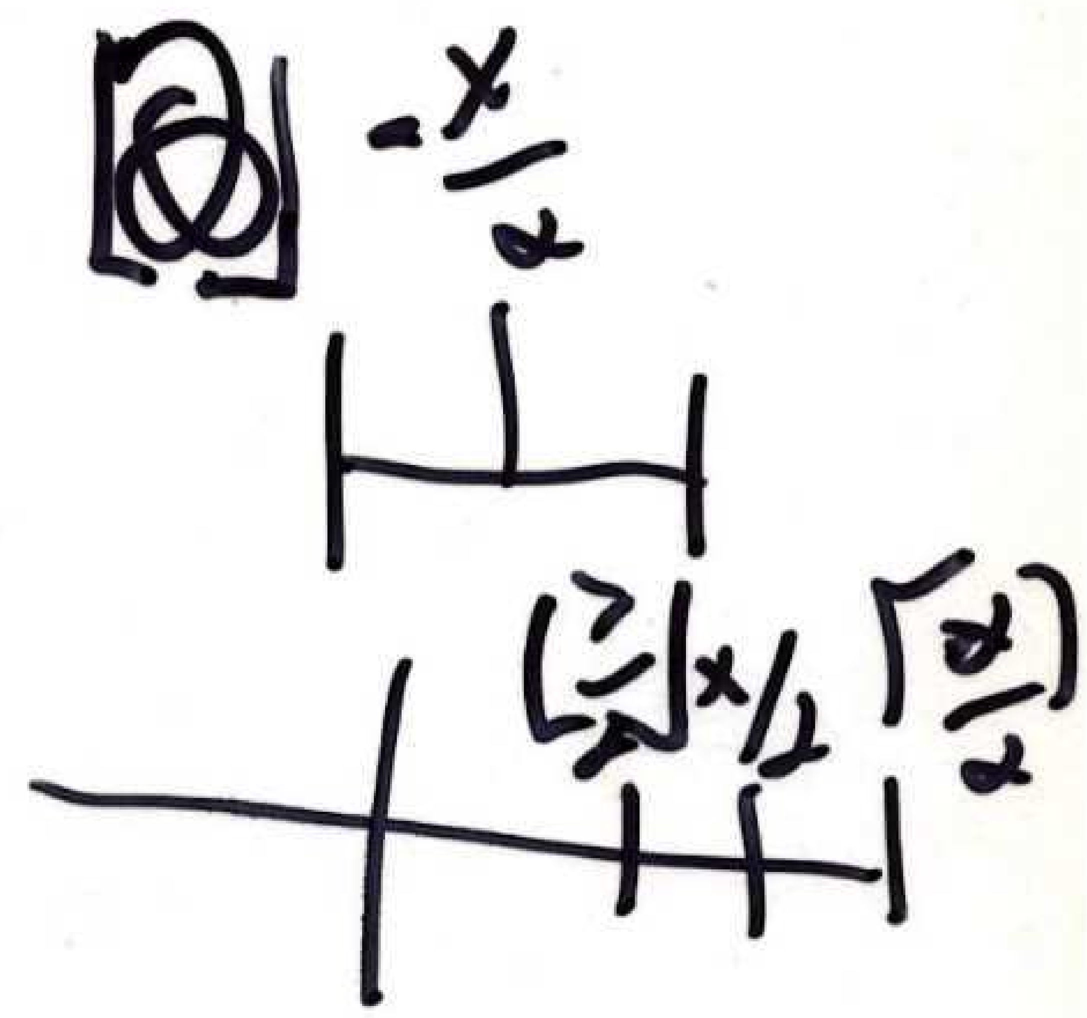
Integer x
Irrational α

$$\frac{x}{\alpha} - \left\lfloor\frac{x}{\alpha}\right\rfloor + \frac{x(\alpha-1)}{\alpha} - \left\lfloor\frac{x(\alpha-1)}{\alpha}\right\rfloor$$

$$= x - \left\lfloor\frac{x}{\alpha}\right\rfloor - \left\lfloor x - \frac{x}{\alpha}\right\rfloor$$

$$= -\left\lfloor\frac{x}{\alpha}\right\rfloor - \left\lfloor -\frac{x}{\alpha}\right\rfloor$$

$$= -\left\lfloor\frac{x}{\alpha}\right\rfloor + \left\lceil\frac{x}{\alpha}\right\rceil$$



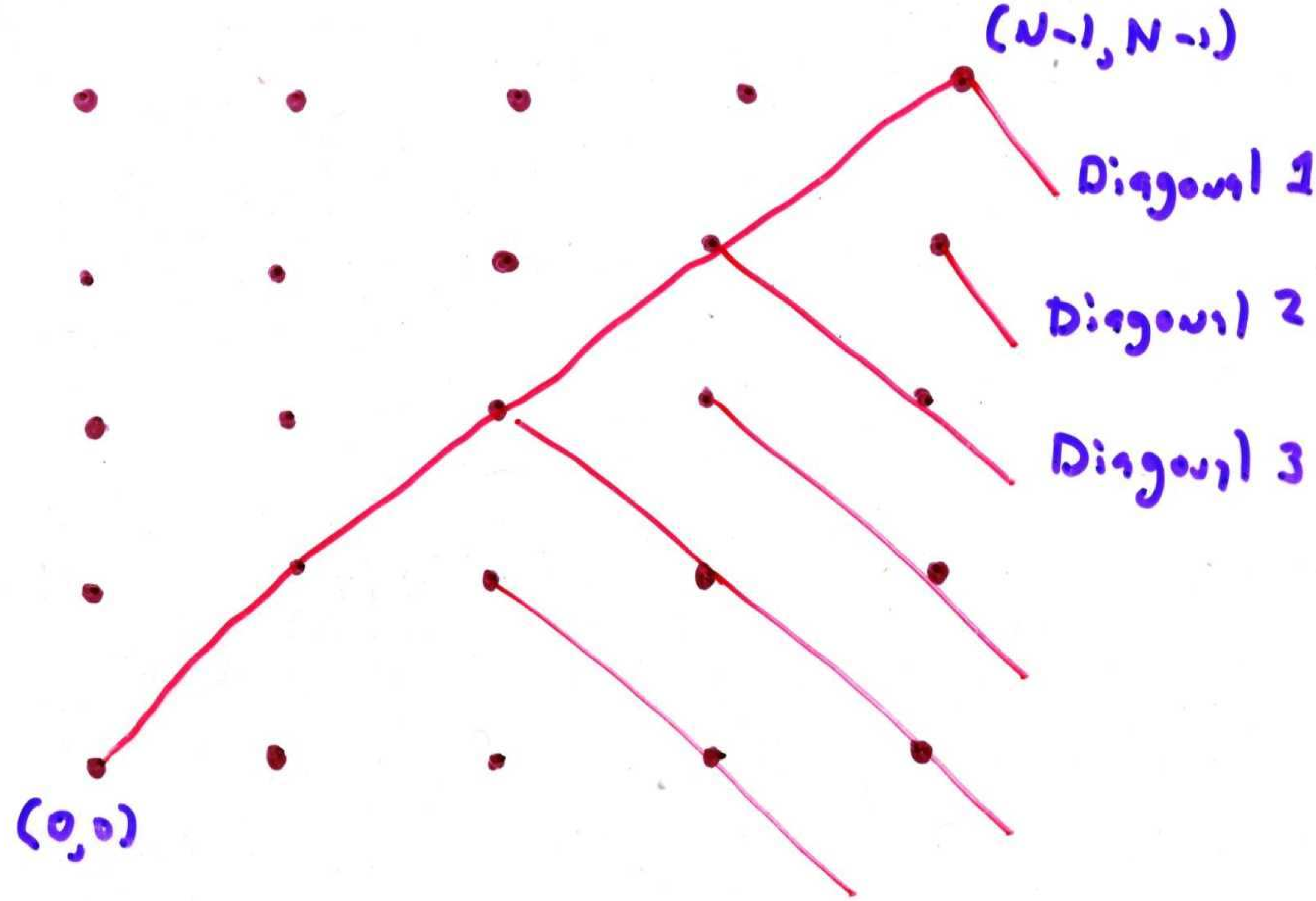
Since $\frac{x}{\alpha}$ is never an integer

$$\left\lceil\frac{x}{\alpha}\right\rceil - \left\lfloor\frac{x}{\alpha}\right\rfloor = 1$$

∎

AMAZING!

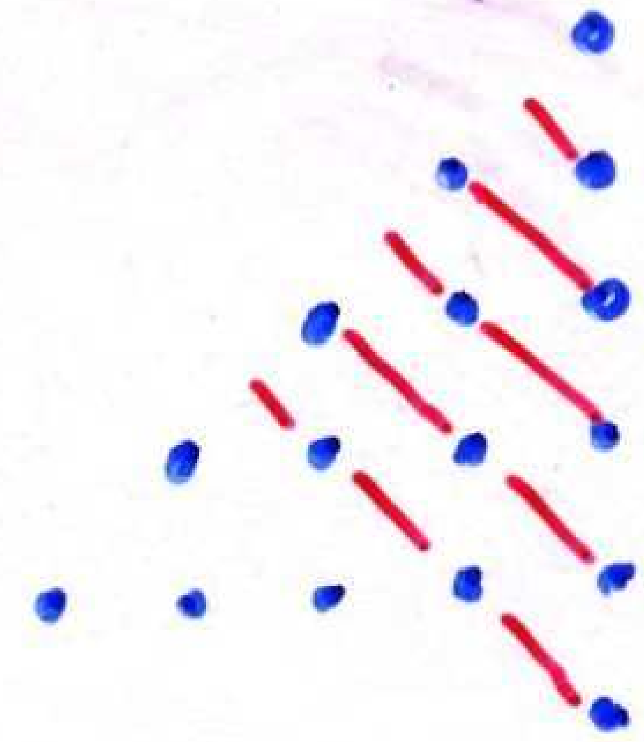
Frequencies of Large Distances in Integer Lattices



Observe that every distinct distance can be described between (0,0) + some point in the lower triangle.

Observe that the distance from (0,0) decreases as we move up a diagonal, since the minimum distance between a point and a line is defined by the perpendicular through the point.

Eventually, the first point on the $d+1^{st}$ diagonal is farther from $(0,0)$ than a point on the d^{th} diagonal



$(0,0)$

But until then, the largest distances are defined by points from bottom to top along increasing diagonals.

Lemma: The i^{th} lattice point is the p_i^{th} point on the d_i^{th} diagonal, where

$$p_i = i - \left\lfloor d_i^2/4 \right\rfloor$$

$$d_i = \left\lceil \sqrt{4i} \right\rceil - 1 = \left\lfloor \sqrt{4i-3} \right\rfloor$$

Observe there are $\lceil d/2 \rceil$ points on the d^{th} diagonal.

Proof of Lemma:

The total # of points on the first $d_i - 1$ diagonals are:

$$\sum_{j} j \left(j \leq \lceil (d_i - 1)/2 \rceil \right) + \sum_{j} j \left(j \leq \lfloor (d_i - 1)/2 \rfloor \right)$$

odd diagonal even diagonals

$$= \frac{1}{2}\left(\left\lceil \frac{d_i - 1}{2} \right\rceil \left\lceil \frac{d_i + 1}{2} \right\rceil + \left\lfloor \frac{d_i - 1}{2} \right\rfloor \left\lfloor \frac{d_i + 1}{2} \right\rfloor \right)$$

if d_i is odd $\Rightarrow \frac{1}{2}\left(\frac{2}{4}(d_i^2 - 1) \right) = \frac{d_i^2 - 1}{4}$

if d_i is even $\Rightarrow \frac{1}{2}\left(\frac{d_i}{2} \cdot \frac{d_i + 2}{2} + \frac{d_i - 2}{2} \frac{d_i}{2} \right) = d_i^2/4$

Thus $i = P_i + \left\lfloor \frac{d_i^2}{4} \right\rfloor$.

Since $1 \leq P_i \leq \lceil d_i/2 \rceil$ (it lies on the d_ith diagonal)

$$\left\lfloor \frac{d_i^2}{4} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{d_i^2}{4} \right\rfloor + \left\lceil \frac{d_i}{2} \right\rceil$$

which can be simplified to

$$d_i = \lceil \sqrt{4i} \rceil - 1 = \lfloor \sqrt{4i - 3} \rfloor$$

But when is the p^{th} point of the d^{th} diagonal farther from 0 than the first point of the $(d+1)$st.?

$$\sqrt{(N-p)^2+(N-d-1+p)^2} > \sqrt{(N-1)^2+(N-d-1)^2}$$

which simplifies to

$$d < \frac{2p(p-1)+2N-1}{2p}$$

but d is an integer + the numerator + denominator are of different parity, so

$$d \le \frac{2p(p-1)+2N-2}{2p} = \frac{N-1}{p}+p-1 \quad (*)$$

Solving for p gives

$$p \le \frac{(d+1)-\sqrt{(d+1)^2-4(N-1)}}{2}$$

Since the last point of the d^{th} diagonal is $\lfloor d/2 \rfloor$, we can set $p = \lfloor d/2 \rfloor$ in $(*)$ to find the last diagonal which completely satisfies the above condition

$$d_L \le \frac{N-1}{\lceil d_L/2 \rceil} + \left(\lceil d_L/2 \rceil - 1 \right)$$

if d_L is even

$$d_L \le \frac{2(N-1)}{d_L} + \frac{d_L}{2} - 1$$

which can be solved as:

$$d_L \le \sqrt{4N-3} - 1$$

if d_L is odd

$$d_L \le \frac{2(N-1)}{d_L+1} + \frac{d_L+1}{2} - 1$$

which can be solved as:

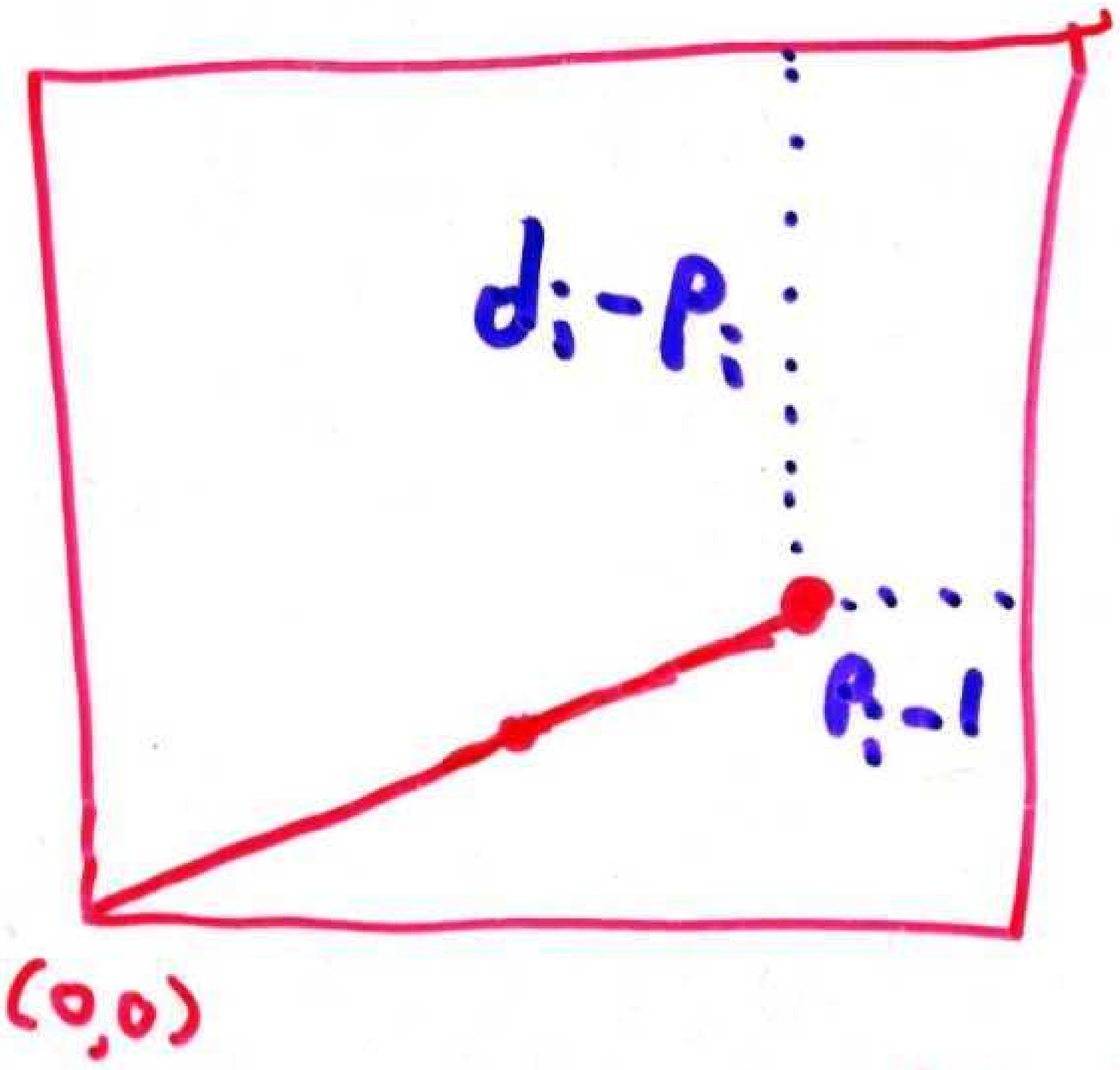
$$d_L \le \sqrt{4N-4} - 1$$

Note $\begin{array}{l} d_L \le \sqrt{4N-3} - 1 \\ d_L \le \sqrt{4N-4} - 1 \end{array} \Rightarrow d_L \le \lfloor \sqrt{4N-3} \rfloor - 1$

since $\lfloor \sqrt{4N-3} \rfloor = \lfloor \sqrt{4N-4} \rfloor$ except when $4N-3$ is a perfect square.

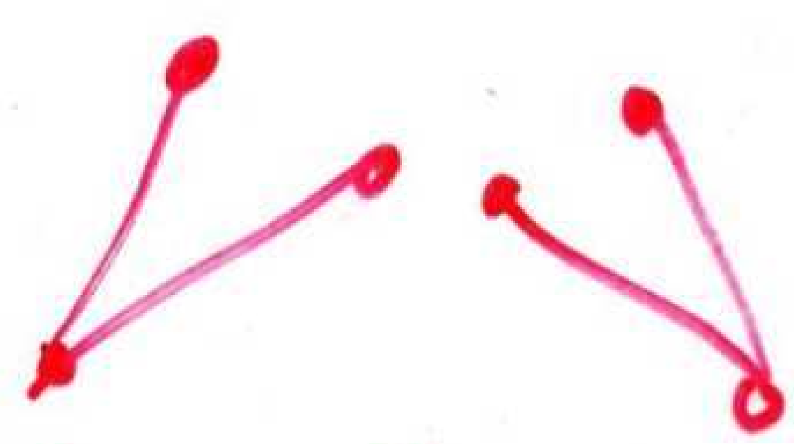
when $4N-3$ is a perfect square, $\sqrt{4N-3} - 1$ is even, and satisfies the first inequality.

Using this, it is possible to show that the first N-1 lattice points define the N-1 largest distances. But what is their frequency?



translations are limited to a box $(d_i - p_i) * (p_i - 1)$

There are 4 distinct reflections



unless the point is on the line $y=x$, when there are two. So the final result is:

$$4\left(i - \left\lfloor \frac{\lceil 2\sqrt{i}\rceil^2}{4} \right\rfloor\right)\left(\lceil 2\sqrt{i}\rceil - i + \left\lfloor \frac{\lfloor 2\sqrt{i}\rfloor^2}{4} \right\rfloor\right)$$

and

$2i$ when i lies on a main diagonal and is a perfect square.