Making Change

How many different ways are there of making change of 50¢ in terms of pennies, nickels, dimes, quarters, + half-dollars.

Veterans of CSE 548 may remember a dynamic programming algorithm for this problem, but we are looking for an analytic solution:

The ways we can make change from pennies is:

$$P = \cancel{1} + ① + ①^2 + ①^3 \dots$$

The ways we can make change from nickels + pennies is:

$$N = P + ⑤P + ⑤^2P + ⑤^3P \dots$$

This is simply restating that for a given number of nickels, the ways to make change depend on penny combinations.

$$D = N + (10)N + (10)^2N + (10)^3N \dots$$

$$Q = D + (25)D + (25)^2D + (25)^3D \dots$$

$$C = Q + (50)Q + (50)^2Q + (50)^3Q \dots$$

C is a generating function containing all ways to make change from the five denominations.

We can reduce this generating function to one which counts the number of ways of making change by making all combinations of the same amount equal.

This can be done by using z as the cash value, so:

$$P = 1 + z + z^2 + z^3 \ldots = \frac{1}{1-z}$$

$$N = P(1 + z^5 + z^{10} + z^{15} \ldots) = P/(1-z^5)$$

$$D = N(1 + z^{10} + z^{20} + \cdots) = N/(1-z^{10})$$

$$Q = D(1 + z^{25} + z^{30} + \cdots) = D/(1-z^{25})$$

$$C = Q(1 + z^{50} + z^{100} \ldots) = Q/(1-z^{50})$$

Thus a generating function for the number of ways to make change is:

$$C = \frac{1}{(1-z)} \cdot \frac{1}{(1-z^5)} \cdot \frac{1}{(1-z^{10})} \cdot \frac{1}{(1-z^{25})} \cdot \frac{1}{(1-z^{50})}$$

We will later see how to expand this.

Keep in mind the key idea – we are letting generating function manipulation do the clever combinatorial reasoning for us, thus minimizing our need for cleverness.

Integer Partitions

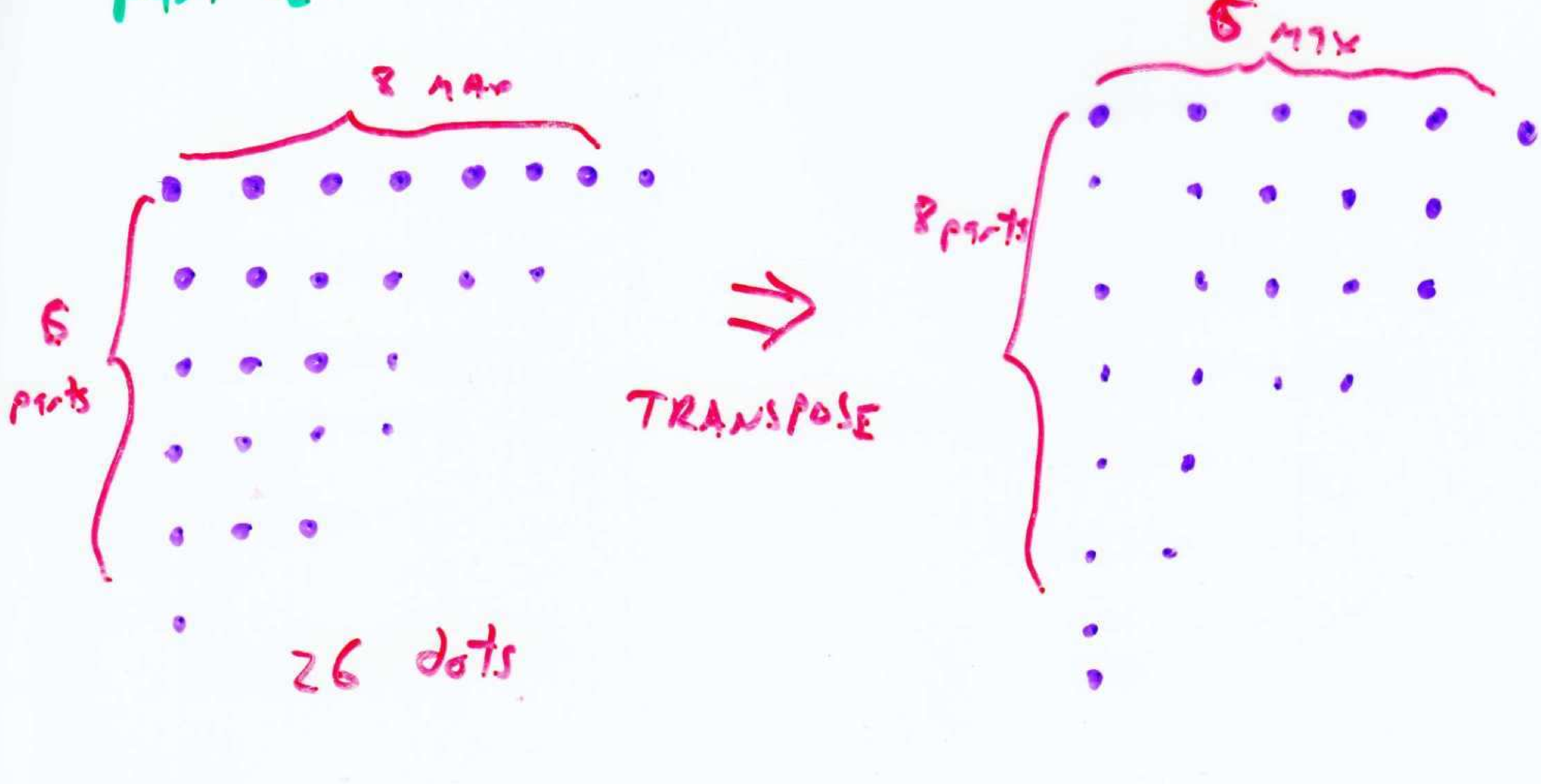
Consider a world with all integer denominations. How many ways are there to partition N?

$N = 6$ $\quad \{6\}, \{5,1\}, \{4,2\}, \{4,1,1\}, \{3,3\},$
$\{3,2,1\}, \{3,1,1,1\}, \{2,2,2\}, \{2,2,1,1\},$
$P(6) = 11$ $\quad \{2,1,1,1,1\}, \{1,1,1,1,1,1\}$

By extending the argument of before,

$$P(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)\cdots}$$

There are many interesting results in the theory of partitions: The number of partitions of N into k things = The number of partitions of N with biggest part k:



Counting Integer Partitions

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$P(N)$	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176

From just this table it is very hard to see how fast this grows.

Hardy & Ramanujan proved that

$$p(N) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dN}\left(\frac{e^{\pi\sqrt{2/3}\,\lambda_N}}{\lambda_N}\right) + O\left(e^{H\sqrt{N}}\right)$$

$$\lambda = \sqrt{N - 1/24}$$

The origin of this preposterous looking result makes vague sense in terms of expanding the generating function

Solving Recurrences - Making Change

Earlier, we came up with this generating function for the number of ways to make change:

$$C(z) = \frac{1}{1-z} \cdot \frac{1}{1-z^5} \cdot \frac{1}{1-z^{10}} \cdot \frac{1}{1-z^{25}} \cdot \frac{1}{1-z^{50}}$$

Since it is a product of rational functions, we can multiply this out, factor it, and then use the canned formulas to give the expanded form and thus the sequence.

That is the right way to do it if we don't mind working with 91^{st} degree polynomials!

Otherwise, we can try to massage it into something more reasonable.

Since $(1-z)(1+z+z^2+z^3+z^4) = (1-z^5)$, we can rewrite things as:

$$C(z) = (1+z+z^2+z^3+z^4)\,\hat{C}(z^5)$$

$$\hat{C}(z) = \left(\frac{1}{1-z}\right)^2 \left(\frac{1}{1-z^2}\right)\left(\frac{1}{1-z^5}\right)\left(\frac{1}{1-z^{10}}\right)$$

So now we only have a 19^{th} degree polynomial.

Block[{k},
S[a_,b_,c_] =   Sum[z^k, {k,a,b,c}]]

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Script started on Mon Nov  2 14:47:13 199
sbskiena% sparcmath
Mathematica 21 for SPARC.
Copyright 1988-92 Wolfram Research, Inc.
 -- Motif graphics initialized --

In[2]:= S[0,50,10]

              10     20     30     40     50
Out[2]= 1 + z   + z   + z   + z   + z

In[3]:= S[0,50,1] * S[0,50,5] * S[0,50,10] * S[0,50,25] * S[0,50,50]

               50           25    50          10     20     30     40     50
Out[3]= (1 + z  ) (1 + z   + z  ) (1 + z   + z   + z   + z   + z  )

             5     10     15     20     25     30     35     40     45     50
>     (1 + z  + z   + z   + z   + z   + z   + z   + z   + z   + z  )

                   2     3     4     5     6     7     8     9     10     11     12     13
>     (1 + z + z  + z  + z  + z  + z  + z  + z  + z  + z   + z   + z   + z   +

        14     15     16     17     18     19     20     21     22     23     24
>      z   + z   + z   + z   + z   + z   + z   + z   + z   + z   + z   +

        25     26     27     28     29     30     31     32     33     34     35
>      z   + z   + z   + z   + z   + z   + z   + z   + z   + z   + z   +

        36     37     38     39     40     41     42     43     44     45     46
>      z   + z   + z   + z   + z   + z   + z   + z   + z   + z   + z   +

        47     48     49     50
>      z   + z   + z   + z  )

In[4]:= Expand[%]

                   2    3    4      5      6      7      8      9      10
Out[4]= 1 + z + z  + z  + z  + 2 z  + 2 z  + 2 z  + 2 z  + 2 z  + 4 z   +

         11       12       13       14       15       16       17       18       19
>     4 z   + 4 z   + 4 z   + 4 z   + 6 z   + 6 z   + 6 z   + 6 z   + 6 z   +

         20       21       22       23       24        25        26        27
>     9 z   + 9 z   + 9 z   + 9 z   + 9 z   + 13 z   + 13 z   + 13 z   +

          28        29        30        31        32        33        34        35
>     13 z   + 13 z   + 18 z   + 18 z   + 18 z   + 18 z   + 18 z   + 24 z   +

          36        37        38        39        40        41        42        43
>     24 z   + 24 z   + 24 z   + 24 z   + 31 z   + 31 z   + 31 z   + 31 z   +

          44        45        46        47        48        49        50        51
>     31 z   + 39 z   + 39 z   + 39 z   + 39 z   + 39 z   + 50 z   + 49 z   +

          52        53        54        55        56        57        58        59
>     49 z   + 49 z   + 49 z   + 60 z   + 59 z   + 59 z   + 59 z   + 59 z   +

          60        61        62        63        64        65        66        67
>     72 z   + 70 z   + 70 z   + 70 z   + 70 z   + 83 z   + 81 z   + 81 z   +
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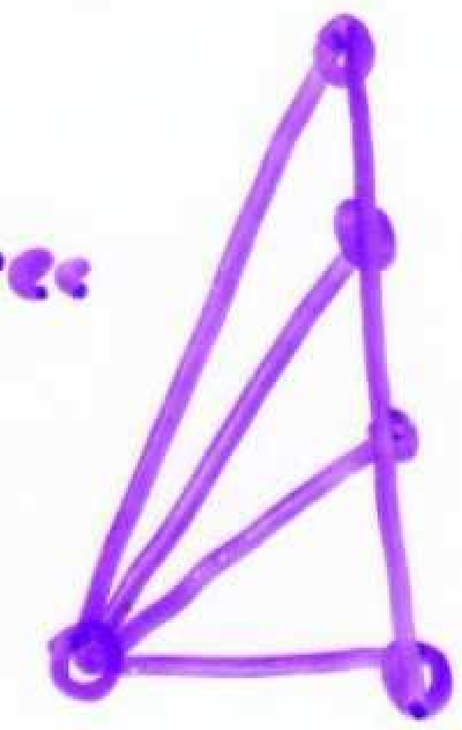
Spanning Trees of Fans

We will get to graph theory in earnest in a few weeks, but first a problem in graphical enumeration.

A *fan* is a graph of $N+1$ vertices & $2N-1$ edges and a degree sequence of $(N, 3, 3, 3 \ldots, 2, 2)$ such that the two vertices of degree 2 are not adjacent.

4 degree fan



A *tree* is a connected graph of $N-1$ edges.

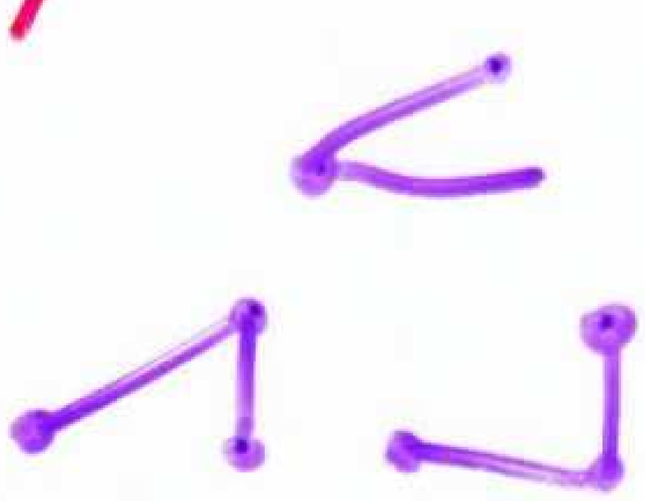
Baseball fan

A *spanning tree* is a subgraph of a graph which is a tree.

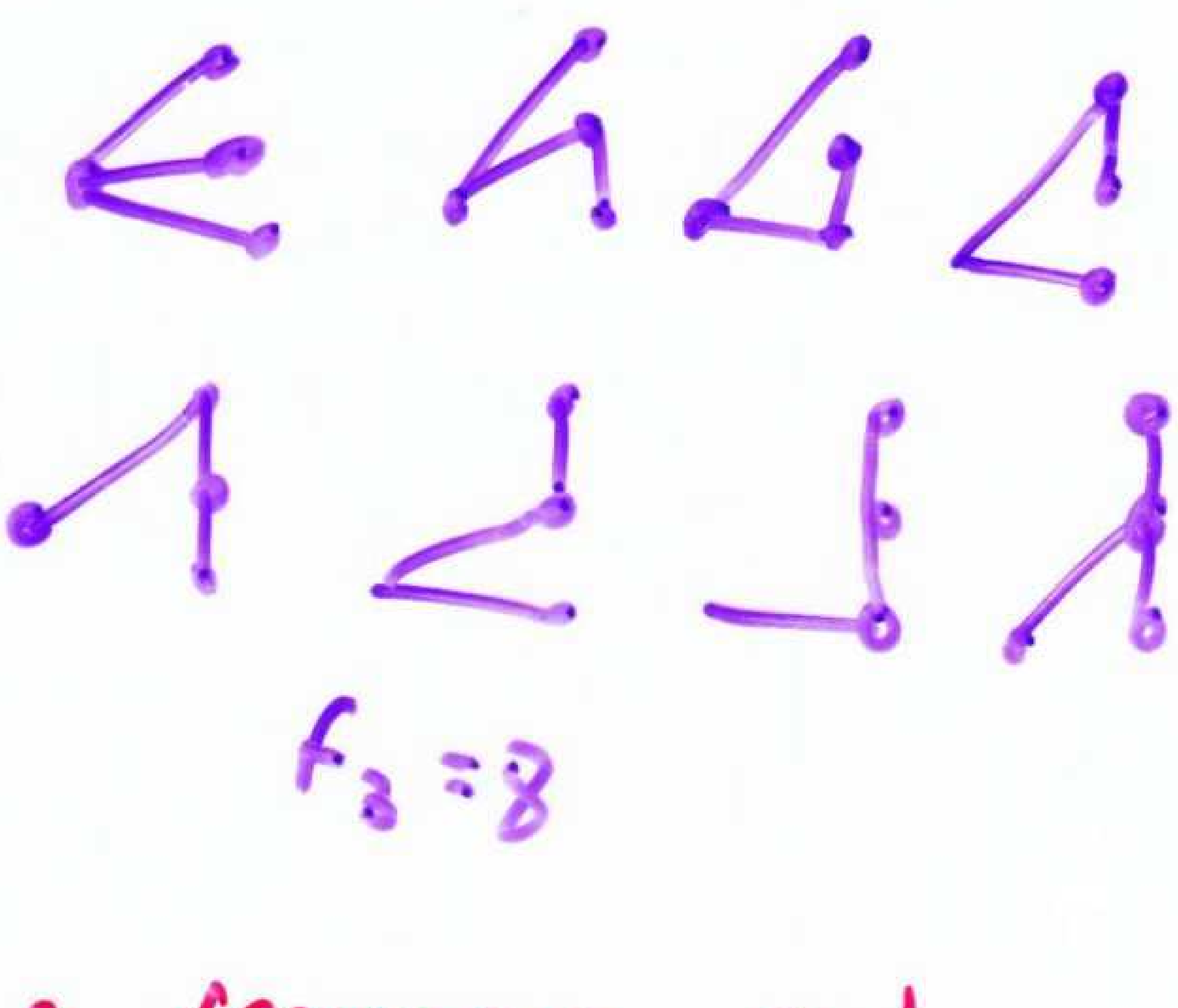
How many spanning trees does a fan have?

$f_1 = 1$

$f_0 = 0$ (-1 edges)



$f_2 = 3$

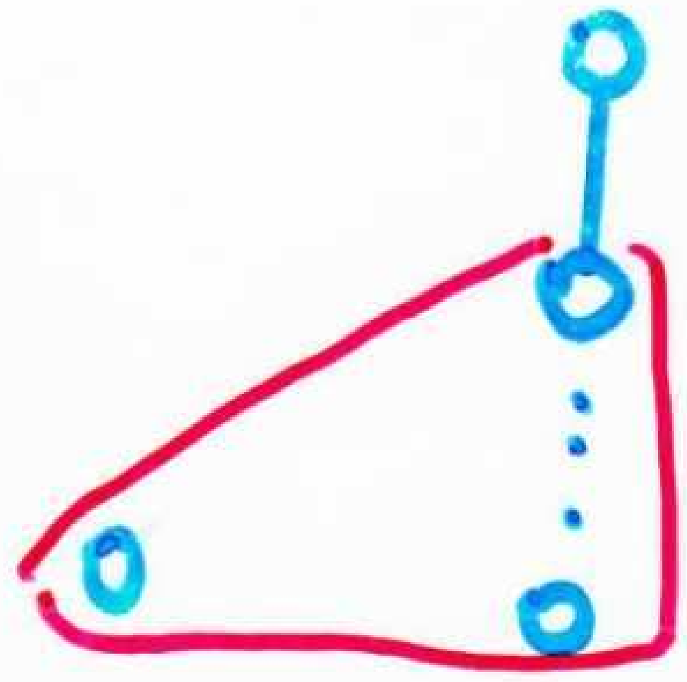


$f_3 = 8$

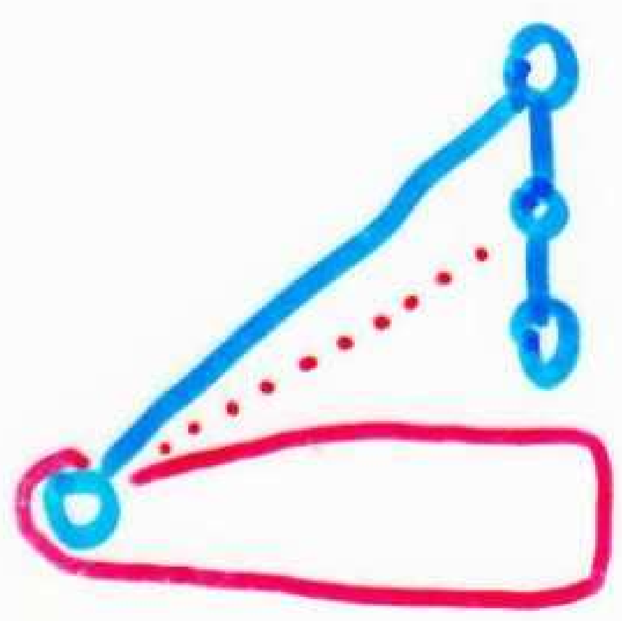
We need to formulate a recurrence and solve it.

We will recur on the number of vertices.

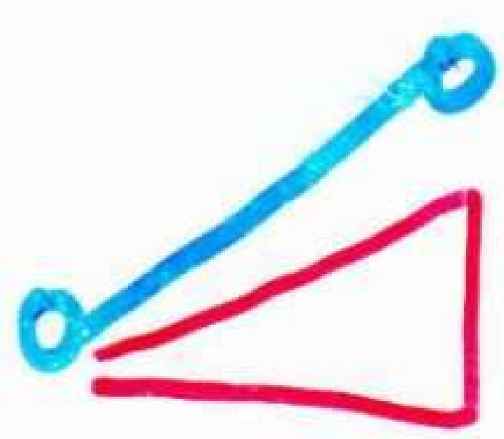
Consider the top vertex of the fan:



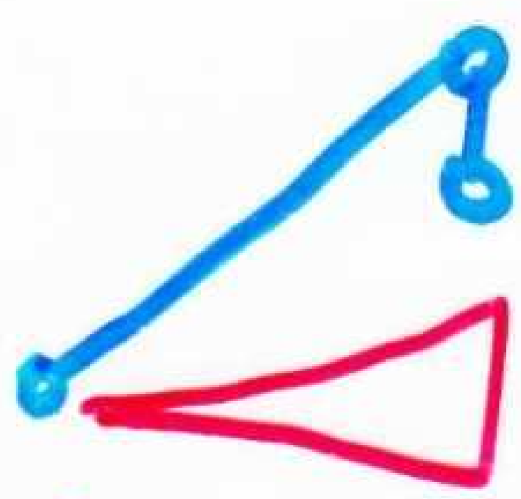
if there is no diagonal edge, the top is connected to its neighbor, and there are f_{N-1} ways to finish.



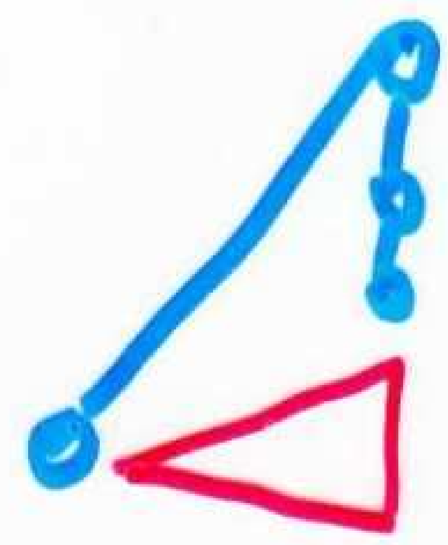
if there is a diagonal edge to the top, there cannot be another diagonal edge with the same blade - or it will create a cycle.



f_{N-1} ways to finish



f_{N-2} ways to finish



f_{N-3} ways to finish

. . . .

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Thus:

$$f_N = f_{N-1} + \sum_{k<N} f_k + (N>0)$$

The summation is potentially nasty

We can solve using generating functions

$$F(z) = \sum f_N z^N = \sum_N f_{N-1} z^N + \sum (N>0) z^N$$

$$+ \sum_{k,N} f_k z^N (k<N)$$

right shift

$$= \sum_k f_k z^k \cdot \underbrace{\sum_N z^{N-k}(N>k)}_{\sum_{N-k>0} z^{N-k}} + zF(z) + \frac{z}{1-z}$$

$$= F(z) \cdot \frac{z}{1-z} + z \cdot F(z) + \frac{z}{1-z}$$

$$\therefore \quad F(z) = \frac{z}{1-z} \cdot \frac{1-z}{1-3z+z^2} = \frac{z}{1-3z+z^2}$$

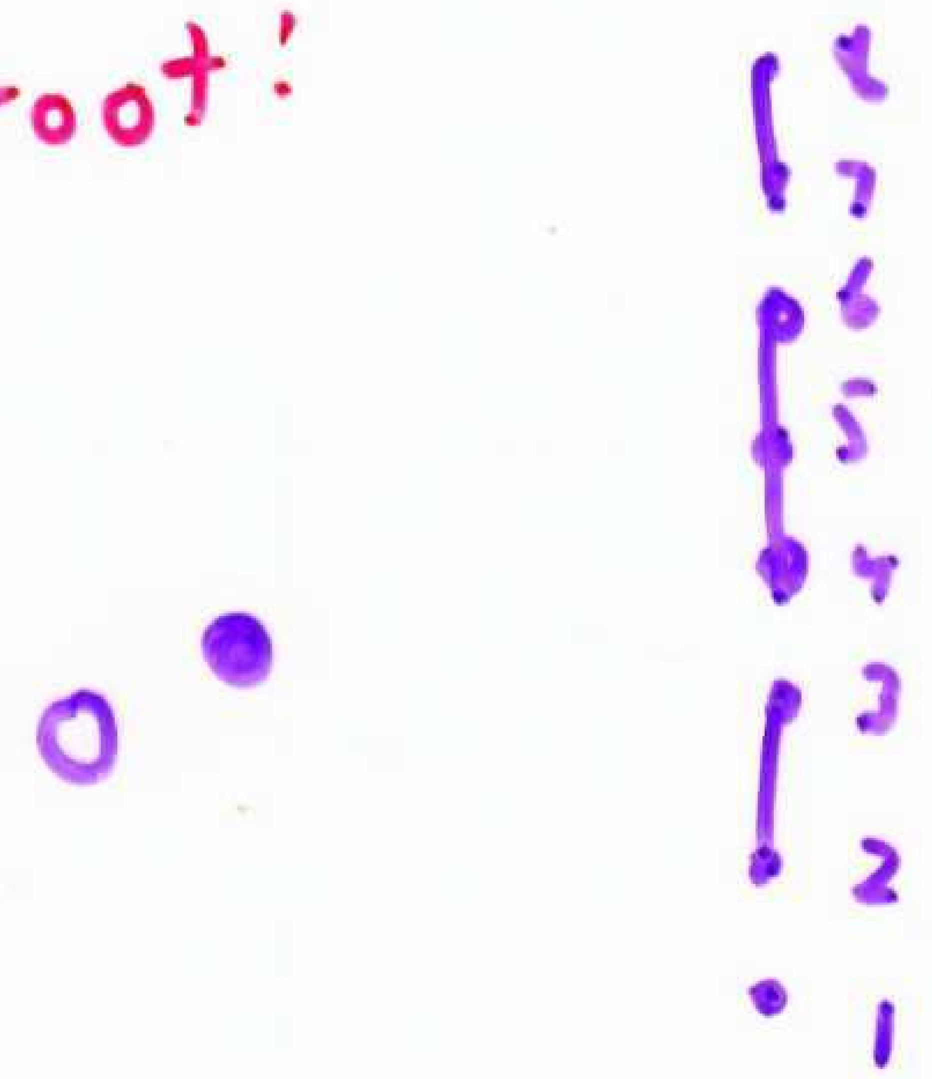
Which can be expanded by rote to give us $f_N = F_{2N}$

The Fibonacci numbers strike again!

Note, setting $f_0 = 1$ would have yielded $\frac{1}{1-3z+z^2}$, only a right shift from what we got.

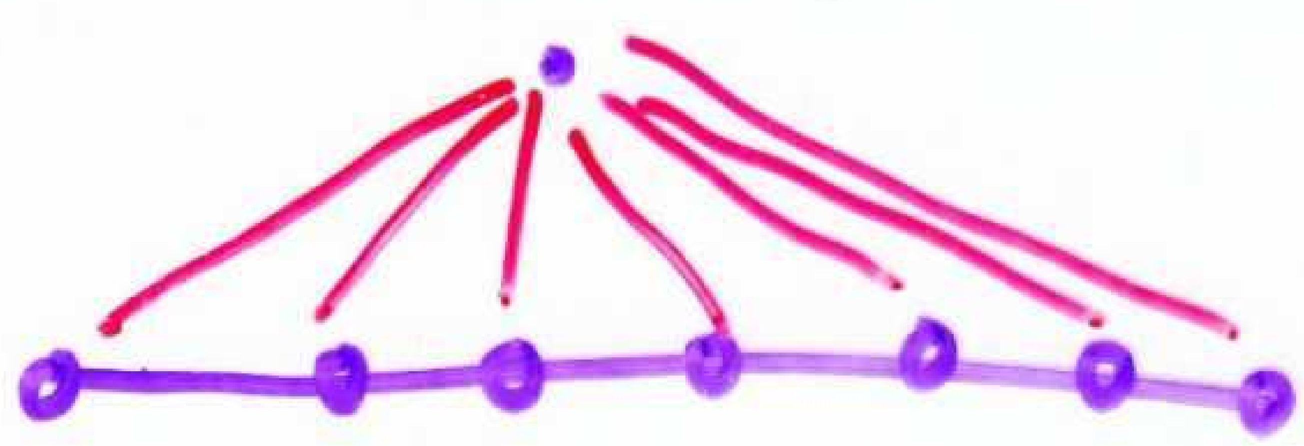
An alternate approach via Convolution

Constructing spanning trees of fans involves two steps - partitioning the N points into connected blades and then attaching it to the root!



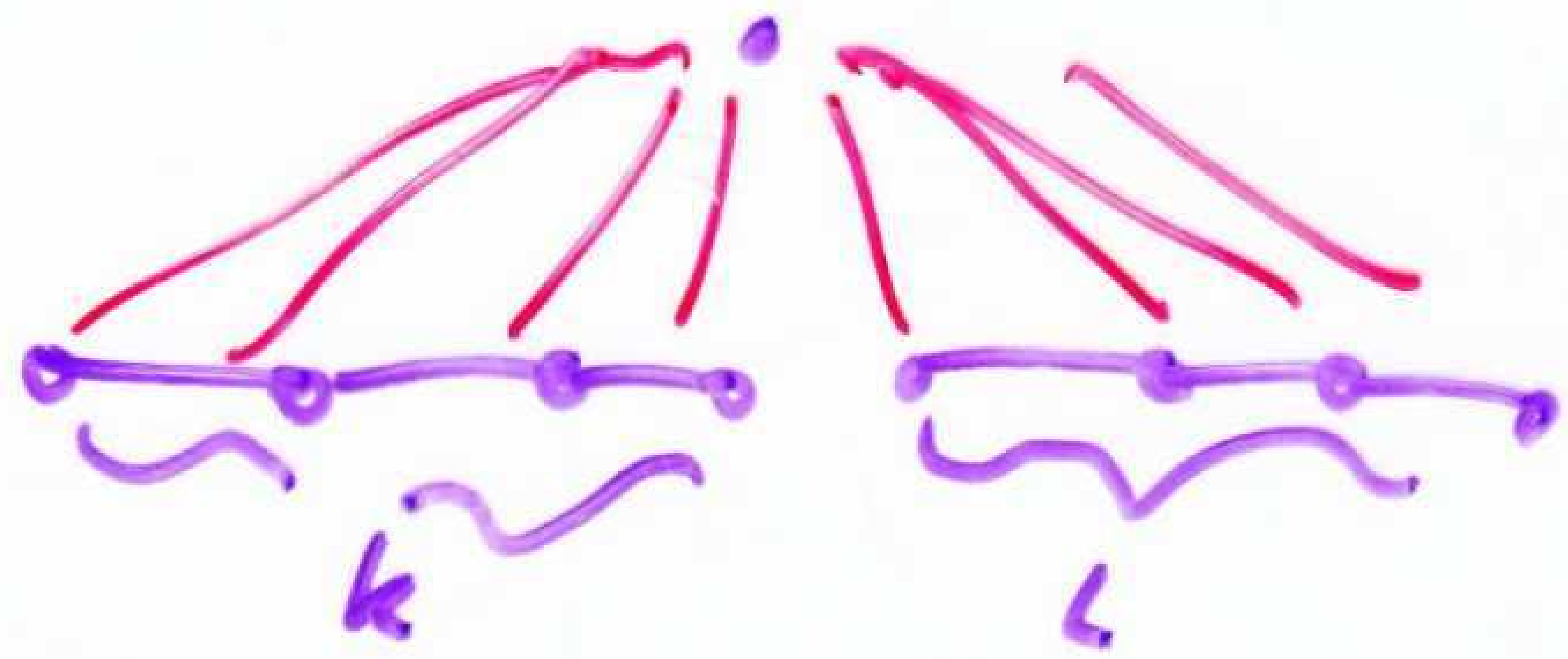
Once we have established our partition connection is easy - any vertex per blade is possible.

Suppose there is one blade - how many trees?



N choices for the spine, one node.

Suppose there are two blades?



k·L choices for the two spines

must iterate through all $k+L=N$

This is a convolution of the sequence $\{0,1,2,3\ldots\}$ with itself!

In general, we have an M-fold convolution:

$$f_N = \sum_{M>0} \sum_{\substack{k_1+k_2\cdots k_M = N \\ k_i > 0}} k_1 k_2 \cdots k_M$$

number of parts

total constraint

ways of connecting them

To convolve functions, multiply the generating functions.

Thus, the generating function is:

$$F(z) = G(z) + G^2(z) + G^3(z) \cdots = \frac{G(z)}{1-G(z)}$$

$$G(z) = \{0, 1, 2, 3 \ldots\ldots\} = \frac{z}{(1-z)^2}$$

So:

$$F(z) = \frac{z}{(1-z)^2 - z} = \frac{z}{1-3z+z^2}$$

Generating functions were introduced to eliminate the need to be clever, but here we have a clever application of generating functions!