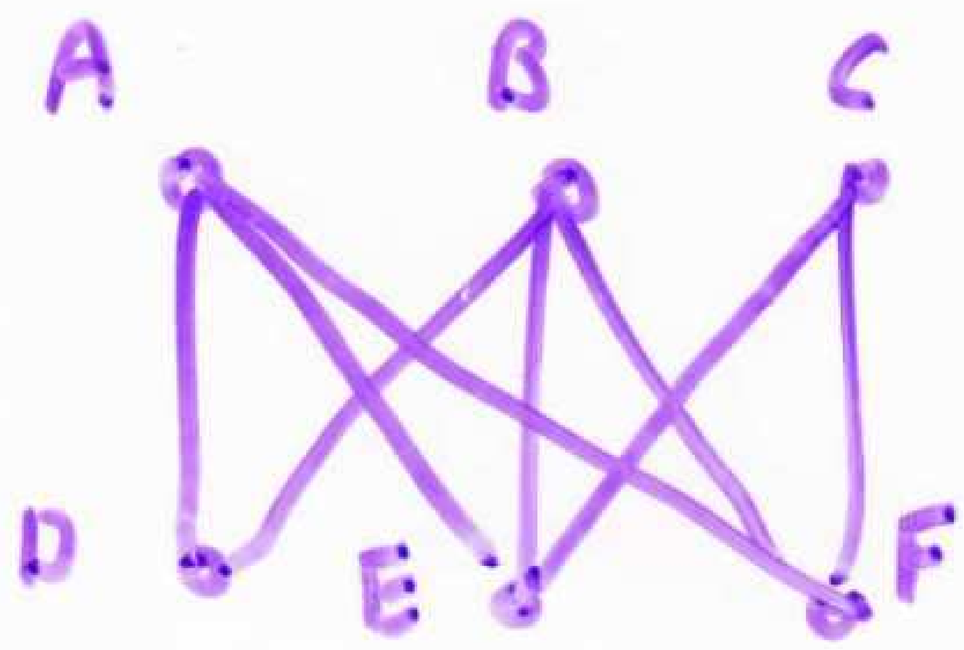
Planar Graphs

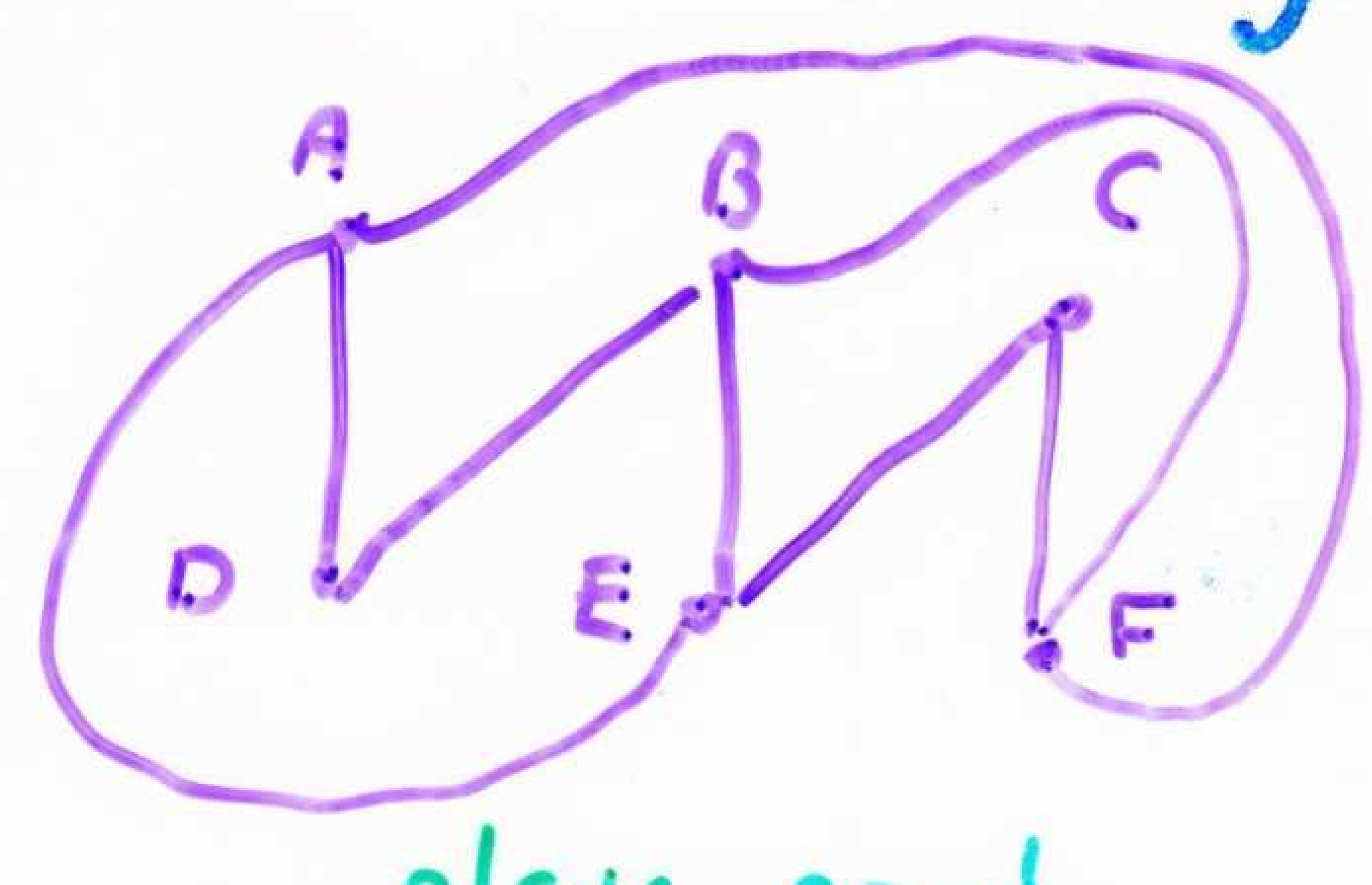
Although graphs are purely combinatorial objects, it is hard to separate our visualization, the drawing, from the underlying structure.

A graph G is **planar** if it can be drawn in the plane with no two edges crossing except at the vertex positions.

Such a drawing is an **embedding** of G, any graph with its embedding is called a **plane graph**



plan<u>a</u>r graph



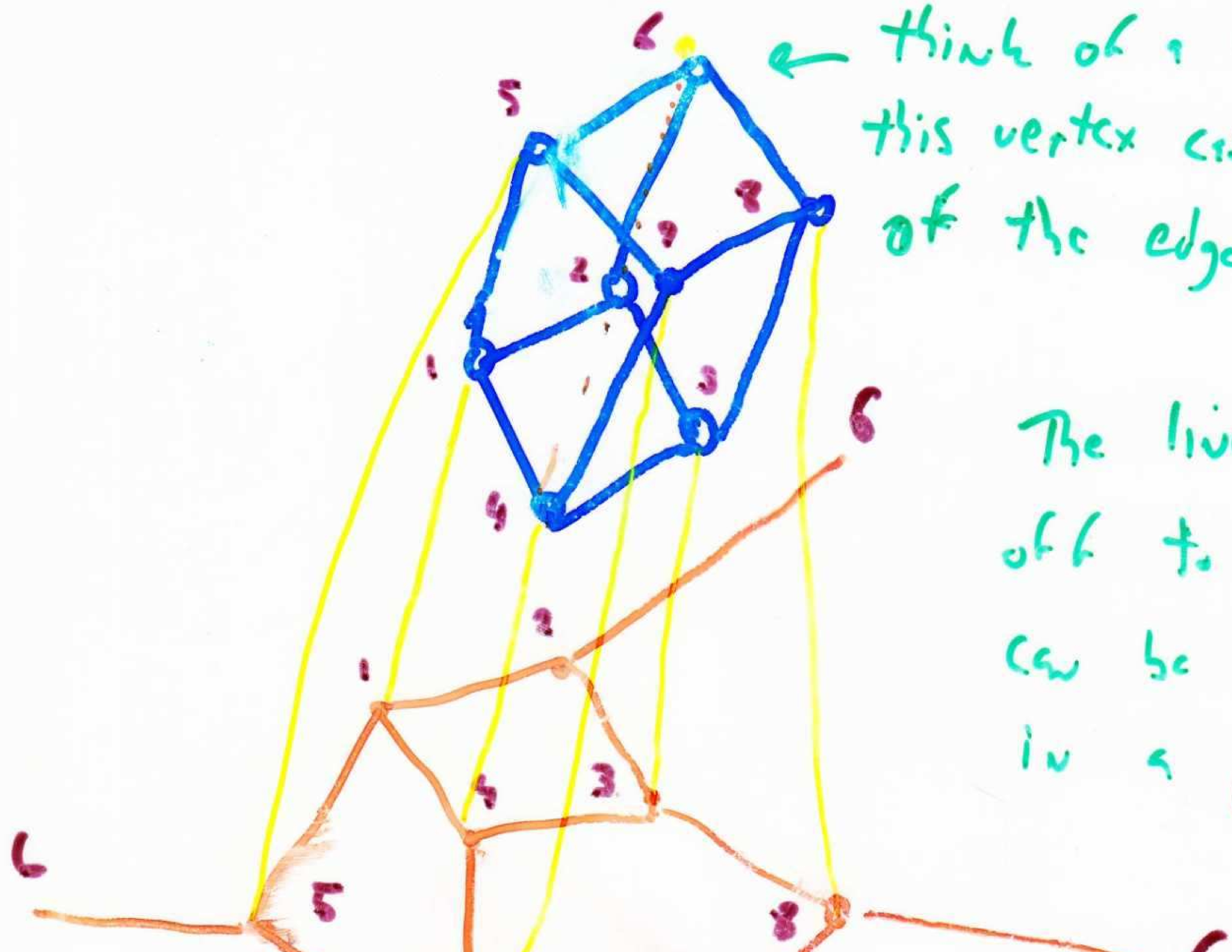
plan<u>e</u> graph

There is a large amount of freedom in how we draw plane graphs.
For example, edge AF could be brought around AE.

"The Slums of Topology"

Whitehead once called graph theory this because he thought planarity was really a topological problem.

Embedding graphs in the plane (minus a point) is the same as embedding them on a sphere. Equivalently, every convex polyhedra defines a graph (same vertices + edges less geometry) which is planar:

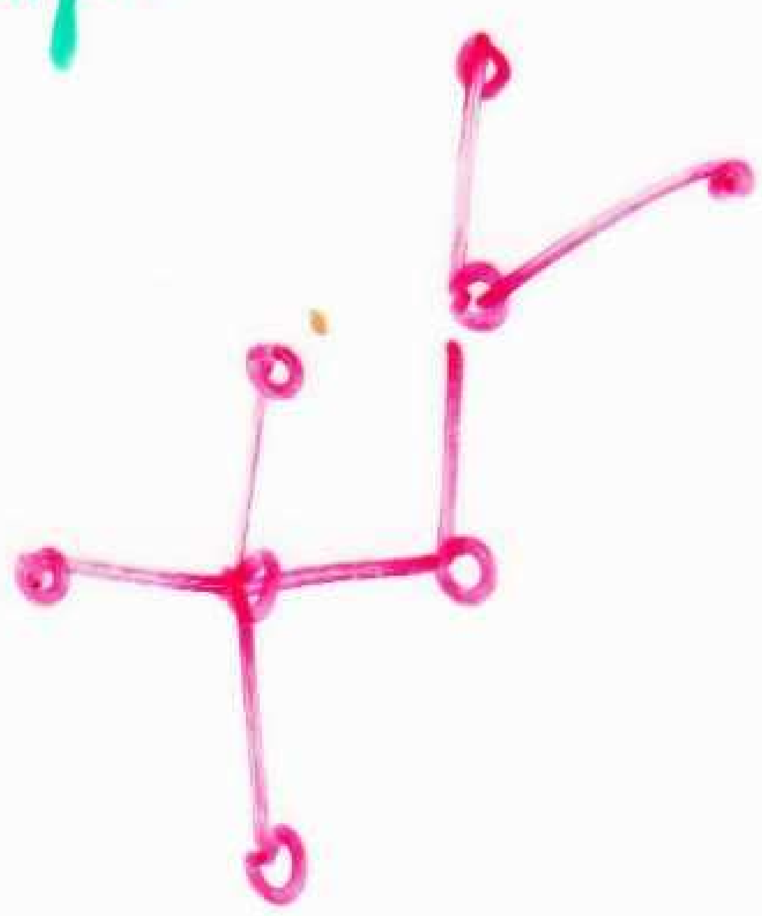


← think of a light at this vertex casting shadows of the edges

The lines going off to infinity can be connected in a point

Euler's Formula

There is a beautiful relation between the number of vertices, edges, and regions of a planar graph: $V - E + R = 2$



tree $\Rightarrow$ $V = N$

$E = N - 1$

$R = \underline{1}$

$N - (N-1) + 1 = 2$

cycle $\Rightarrow$ $V = N$

$E = N$

$R = 2$

$N - N + 2 = 2$

By convention, for construction such as the previous page, we always count the outside region

☆ Note that this implies that the number of regions of any planar graph is <u>independent</u> to the embedding!

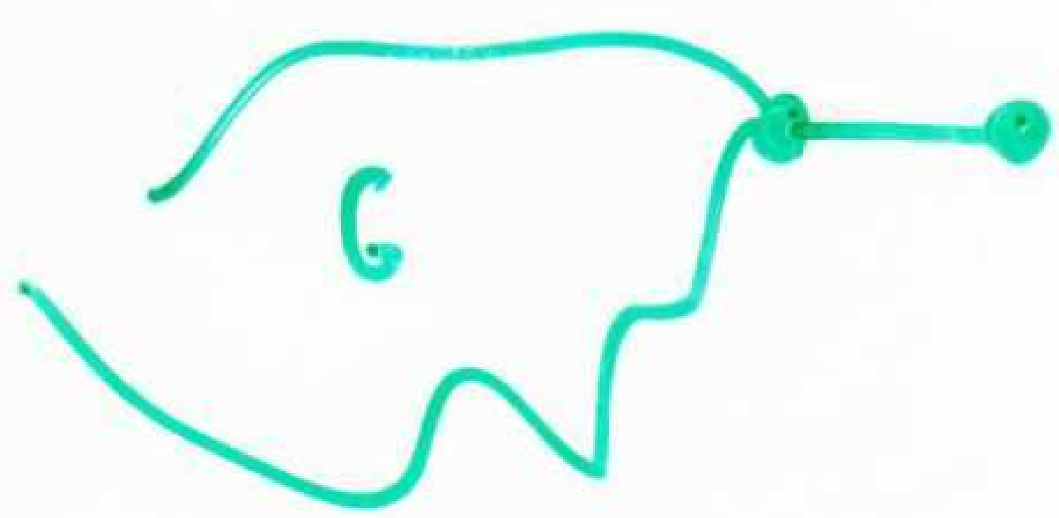
Theorem: For any planar graph, $V - E + R = 2$

Proof: By induction on the number of regions.

As shown before, it is true for trees, $r = 1$.

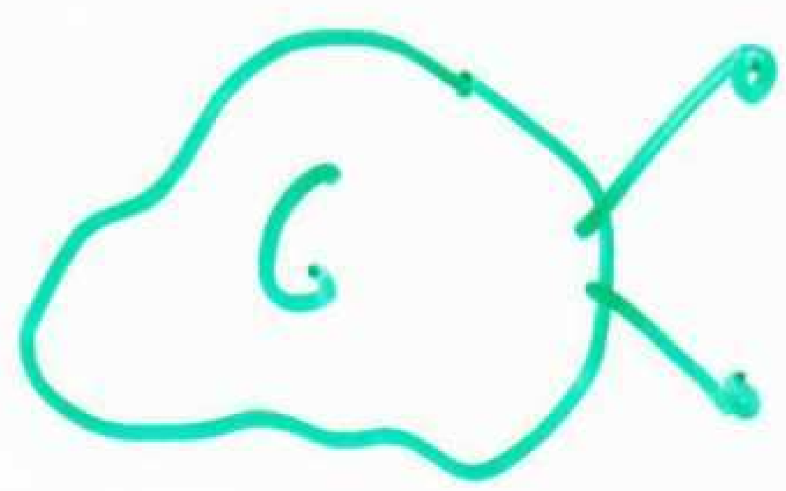
Any plane graph with $< r$ regions satisfies the expression (by assumption)

Any plane graph with r regions can be made by adding vertices and edges to one with $r-1$ regions:



Adding a vertex and one edge without making a region

$(V+1) - (E+1) + R = 2$ ✓



Adding an edge between two existing vertices adds exactly one new region

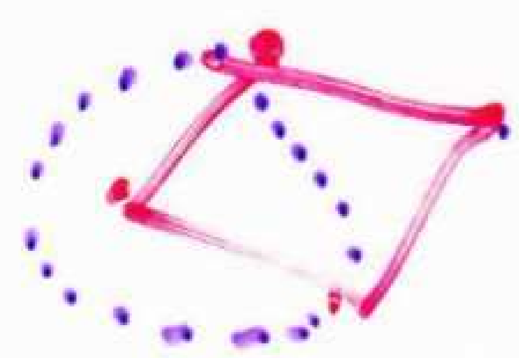
$V - (E+1) + (R+1) = 2$ ✓

$\square$

It is easy to draw graphs in the plane when they have few edges - the more edges, the harder it is:

<u>Theorem</u>: Any planar graph on N vertices contains <u>at most</u> $3N-6$ edges

<u>Proof</u>: For any plan<u>e</u> graph, we can keep adding edges without losing planarity until each face is triangle:



Each edge is incident on exactly <u>two</u> regions and each region is bounded by at least <u>three</u> edges, so $3r \le 2e$ (edge sides)

(at least 3 edge sides per region)

$$\therefore \quad N + r - E = 2$$

$$E = -2 + N + R$$

$$E \le -2 + N + \frac{2E}{3}$$

$$\boxed{E \le 3N - 6}$$

if instead we solve for r:

$$N + r - E = 2$$

$$R = E - N + 2$$

$$R \geq \frac{3R}{2} - N + 2$$

$$\boxed{R \leq 2N - 4}$$

So we get a bound on the maximum value of R...

Further, since $E \leq 3N - 6$, at least one vertex of a planar graph has degree ≤ 5.

Proof: Assume contrary. Since $2E = \sum d_i$,

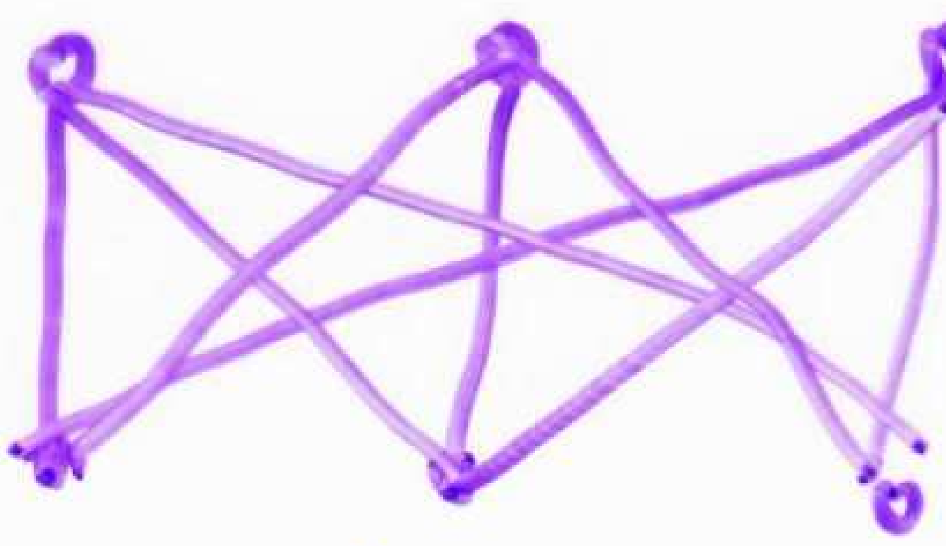
$$2E \geq \sum_{i=1}^{N} 6 \text{ for some graph, so}$$

$E \geq 3N$, a contradiction! $\square$

These properties concerning the sparseness of planar graphs come up over and over.

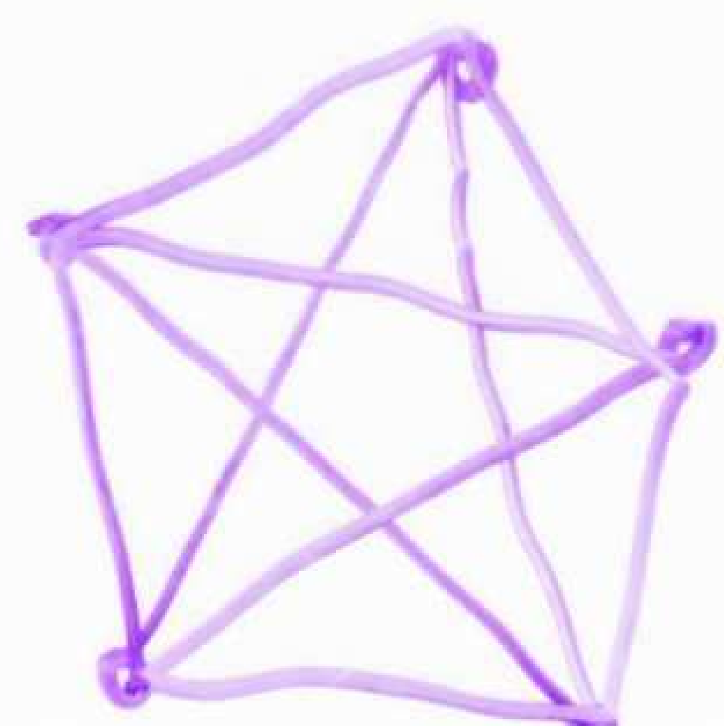
Kuratowski's Theorem

There is a necessary and sufficient condition for planarity. Kuratowski proved that is a graph does not "contain" $K_{3,3}$ or K_5 it is planar:



$K_{3,3}$

trial and error or clever reasoning



K_5

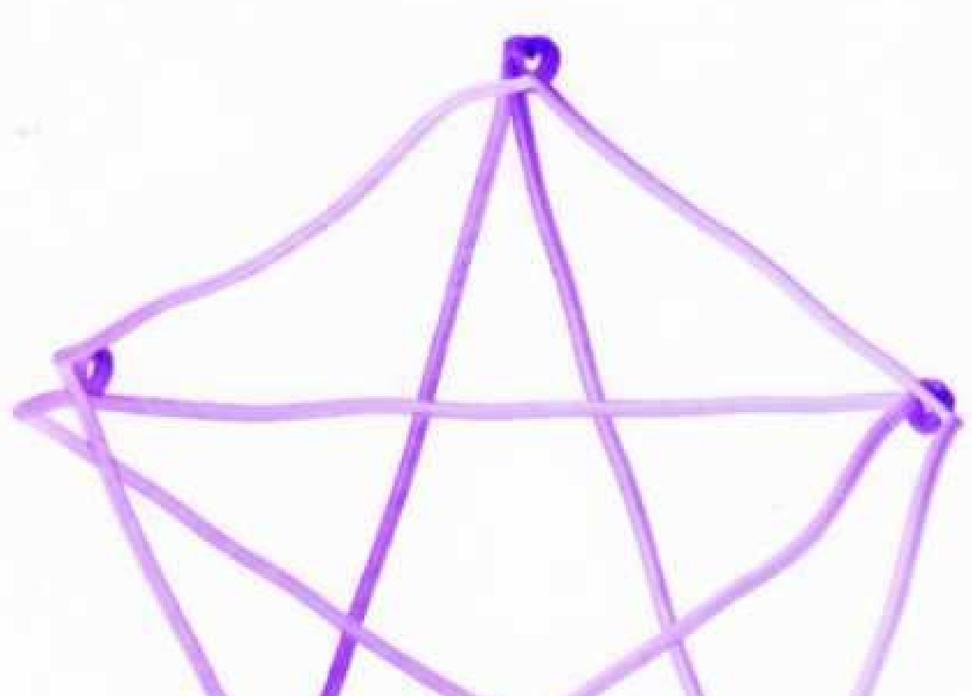
$n = 5$

$e = 10$

$10 \geq 3 \cdot 5 - 6$ X

neither of these are planar.

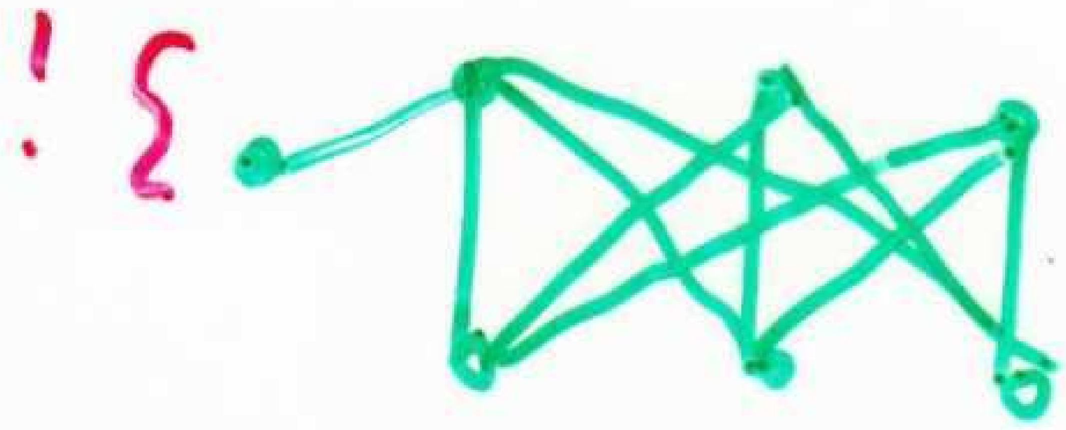
By Kuratowski's theorem, this <u>must</u> be planar:



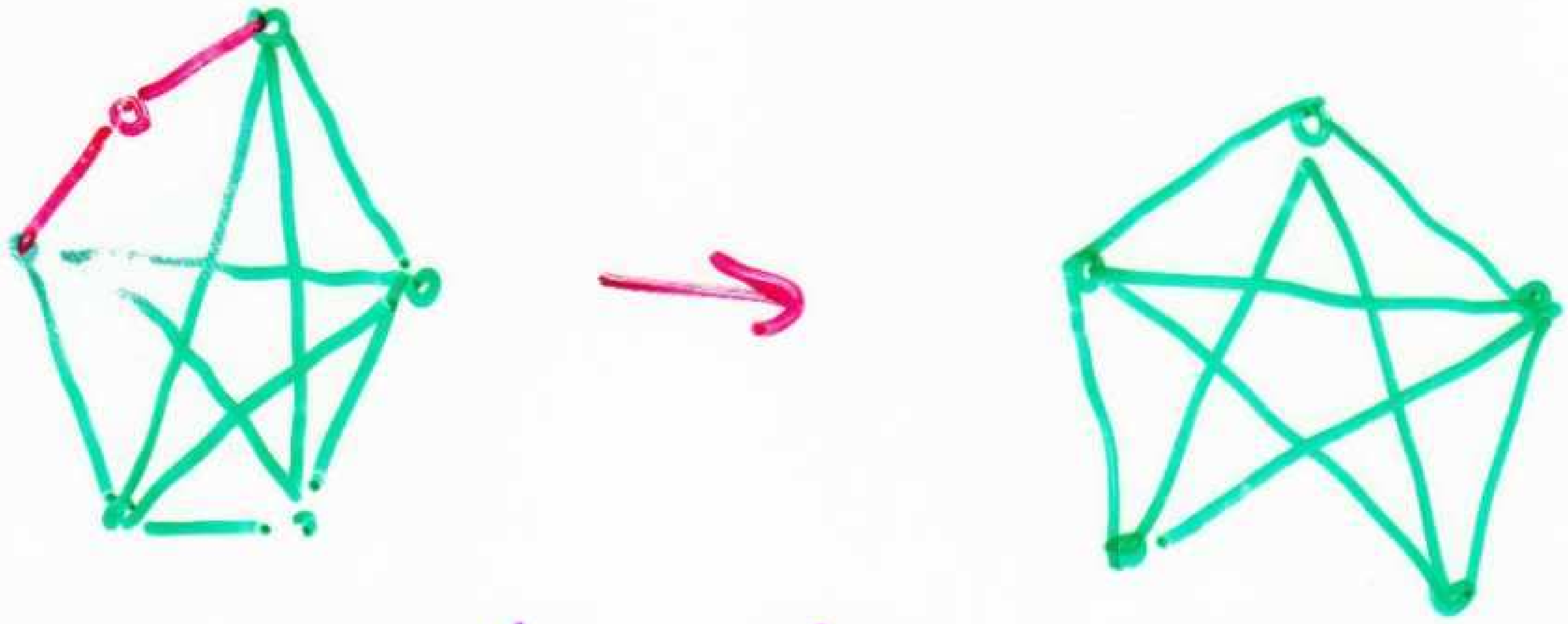
Can you figure out how to embed it?

But what does it mean to "contain"?

Clearly, it is not "isomorphic"



Clearly it is not a set of 5 or 6 vertices which induce the subgraph



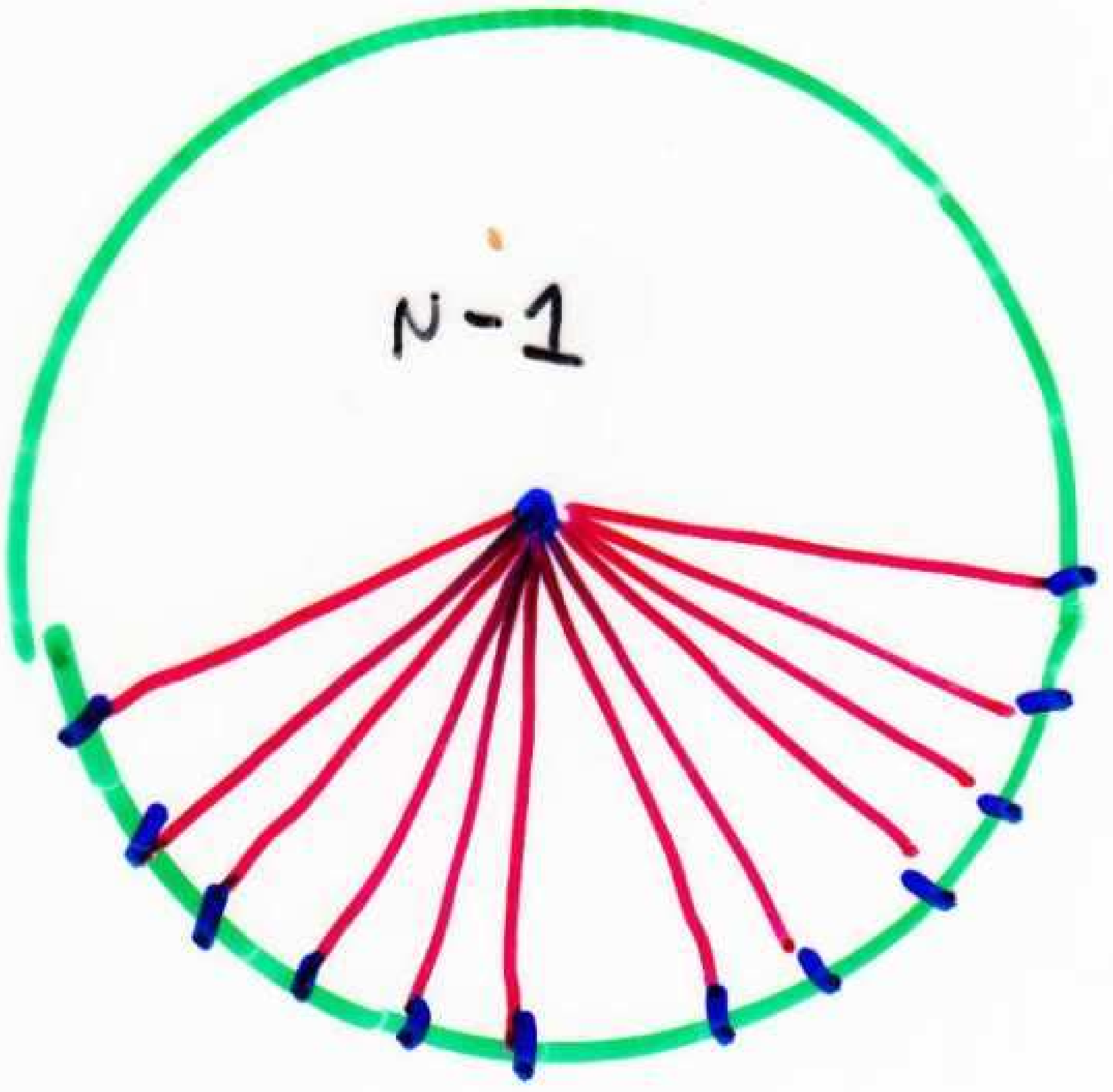
Dividing up the edges with extra vertices cannot effect planarity, since we permit curved edges (which by Fary's theorem are unnecessary anyway!)

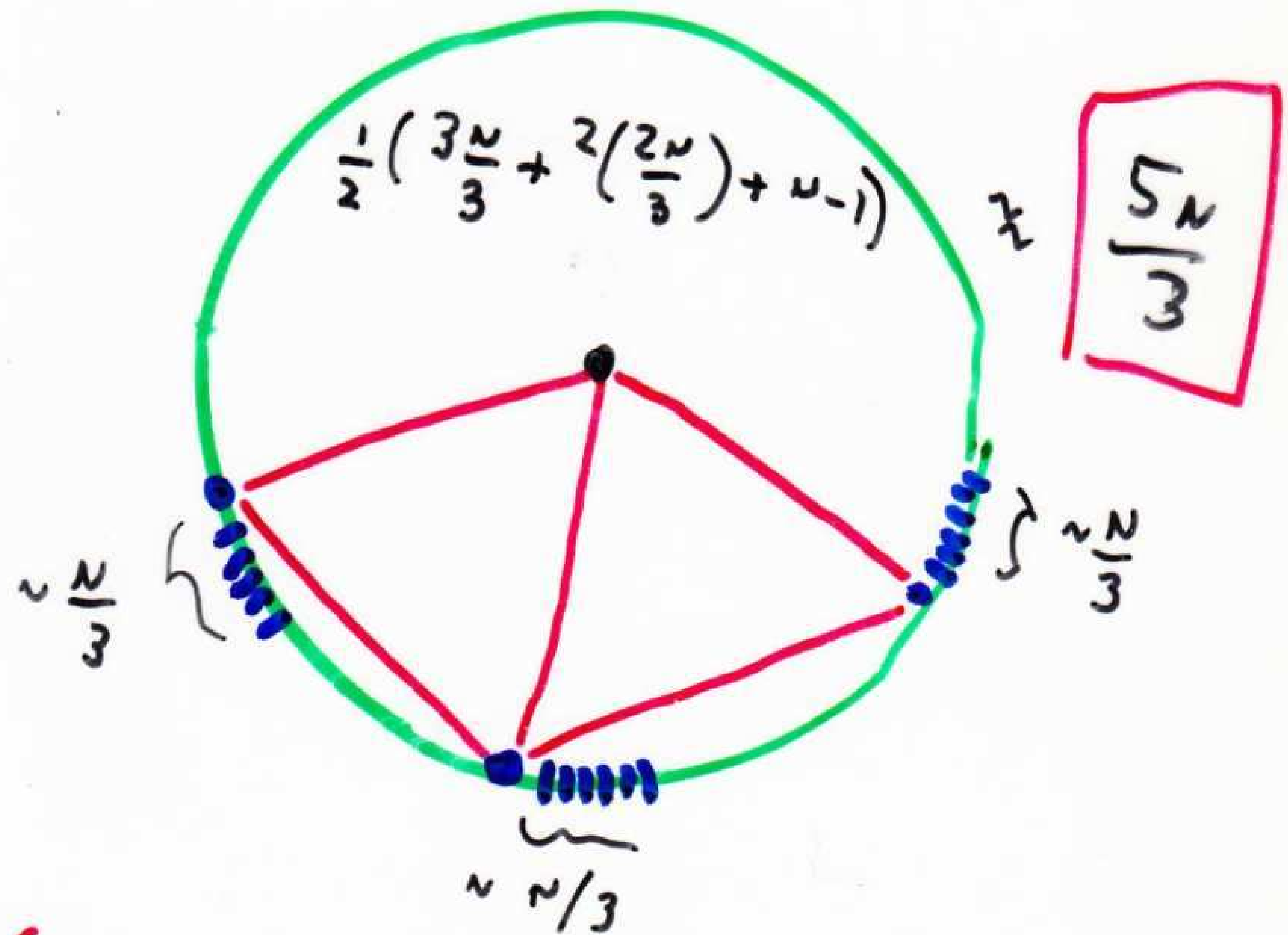
Two graphs are Homeomorphic if they can be obtained from the same graph by dividing up edges.

Kuratowski's Theorem: A graph is planar iff it contains no subgraph homeomorphic to $K_{3,3}$ or K_5!

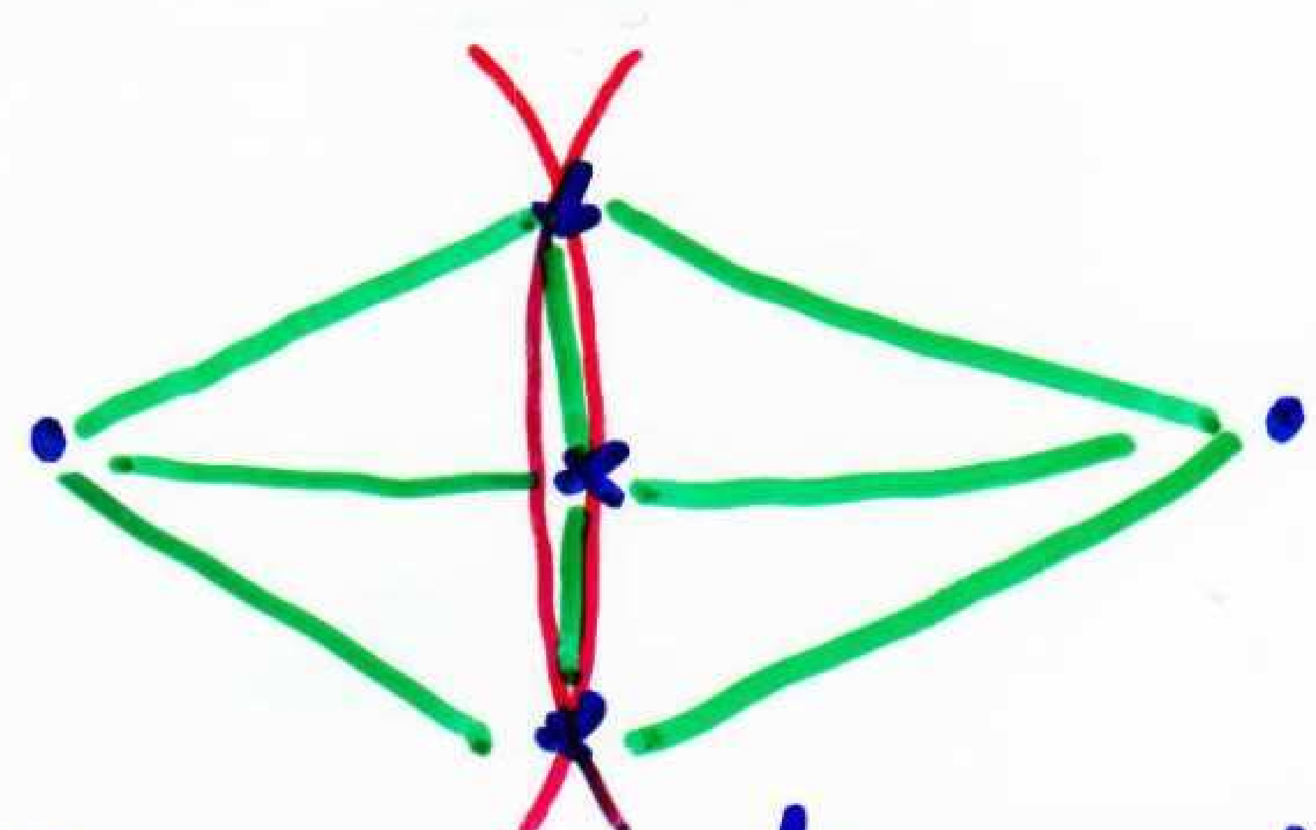
Unit Distances in a Convex Set

For any set of N points in the plane forming a convex set, what is the maximum number of distances of length 1 which can be defined.





Erdös offers \$25 for a proof that there are $O(N)$ unit distances. I once thought I had a simple proof - since $K_{3,3}$ + K_5 are forbidden subgraphs and planar graphs have at most $3N-6$ edges:



Why did my "proof" break

Can't contain $K_{2,3}$ and be convex!