# Matchings

A subset of the edges of a graph form
a **Matching** if no two of the edges are
incident upon the same vertex.
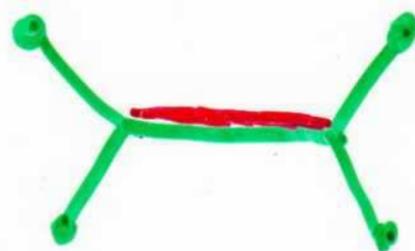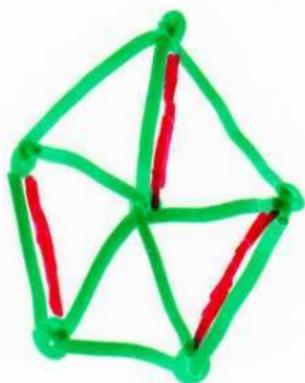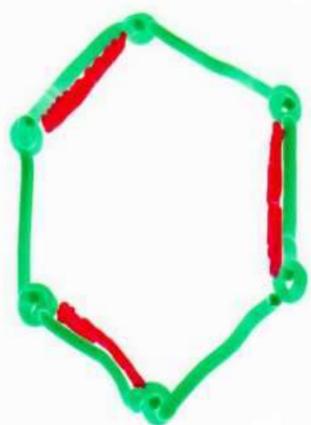
As we will see, Matchings have applications
to a variety of assignment problems.

A **maximal** matching is one which can't
be made larger by adding edges:
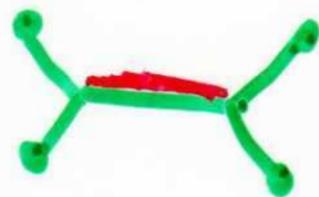
A **Maximum** matching is the largest
possible matching for the graph:
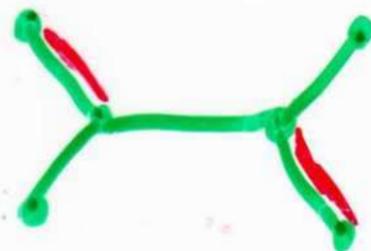
A **perfect** matching contains N
edges in a $2N$ vertex graph:
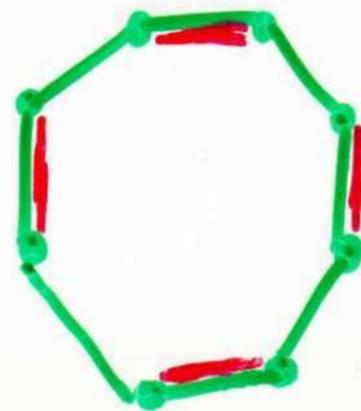
Not all graphs with an even number of vertices have perfect matchings, but many interesting classes of graphs do: cycles, complete graphs, and hypercubes.

$k=1$ $k=2$ $k=3$

$k=4$
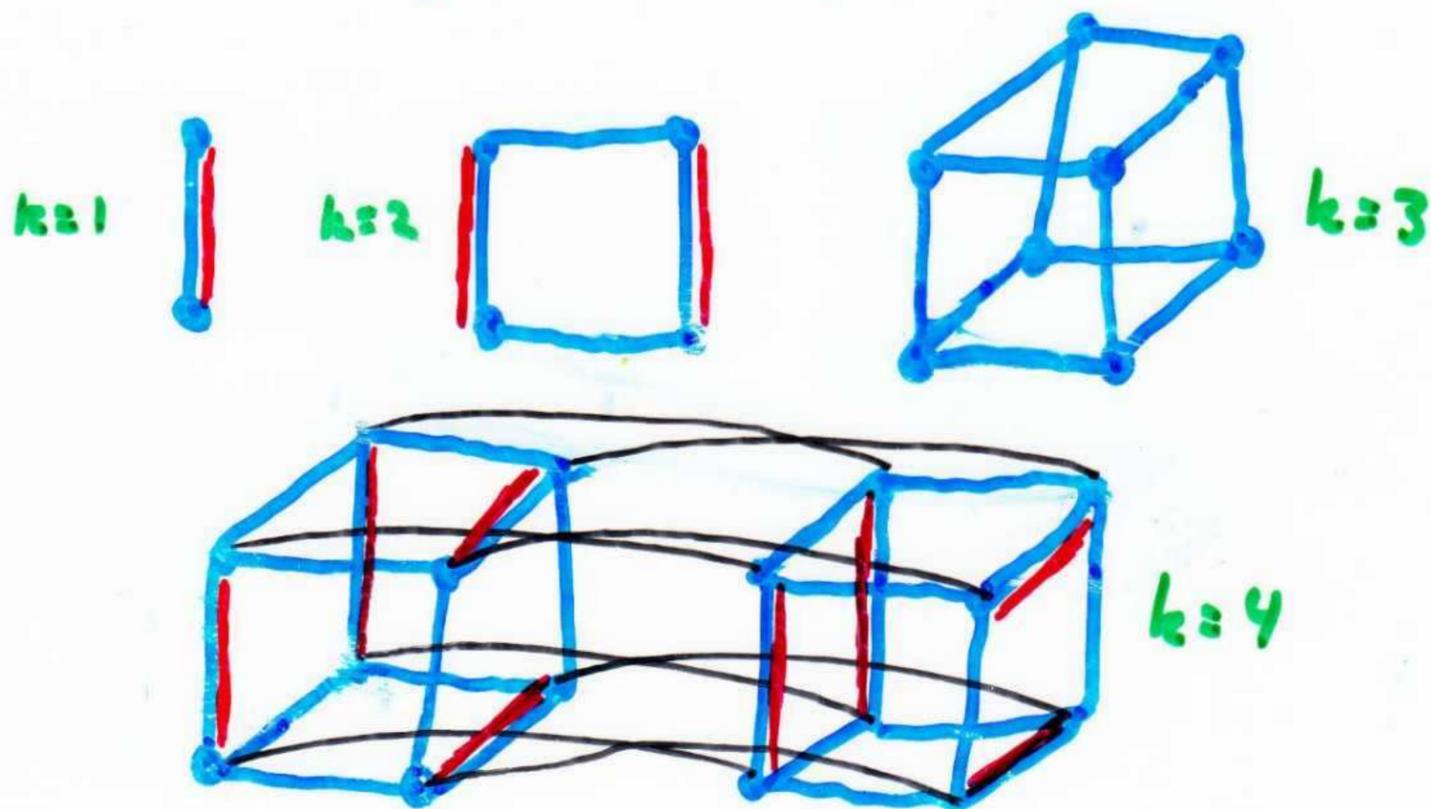
This is just one of the properties which make hypercubes nice. They are also Hamiltonian for $k > 2$, and the distance between any two nodes $\leq k$. Since the diameter is small and the connectivity is high with only $k \cdot 2^{k-1}$ edges, it is a good topology for parallel computers.

Finding a Maximum Matching

There is a fairly simple condition to test if a matching is maximum, and if not, make it larger.

Given a graph G and a matching M, an augmenting path in G is a path of odd length such that the first + last vertices are not in M, and path edges alternate not in M, in M, ... not in M. The path cannot contain a cycle.
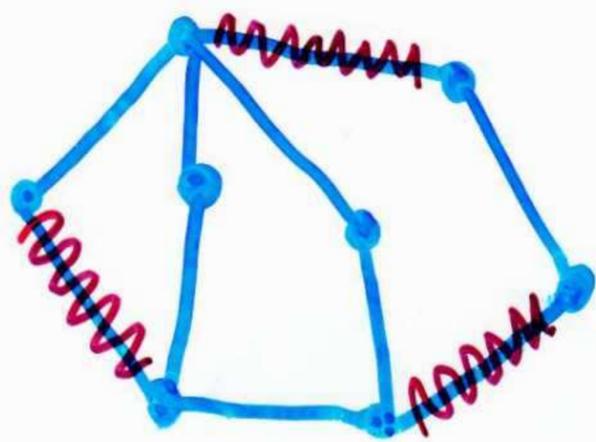
A maximal but not maximum matching
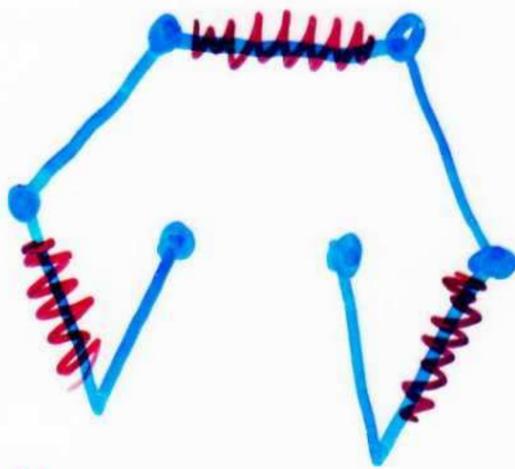
An augmenting path. Starting and ending vertices not in M
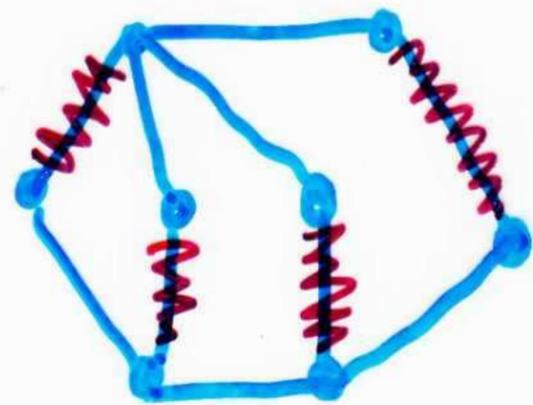
Reversing the edges on this augmenting path makes a larger, in fact perfect, matching!

Clearly, a matching is not maximum if it
contains an augmenting path, since we can
reverse all edges, thus making the matching one
edge larger.

Berge's theorem states this necessary
condition is also sufficient - A matching
is maximum if and only if it contains
no augmenting path!

Proof: Let M' be a maximum matching
of G, and M be non-maximum matching.
Thus: $|M'| > |M|$.

Lets consider the graph formed by the
symmetric difference of M + M' - the
union of the matchings minus the intersection.

Example:

G
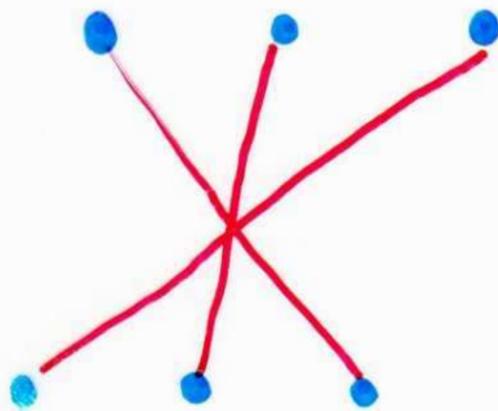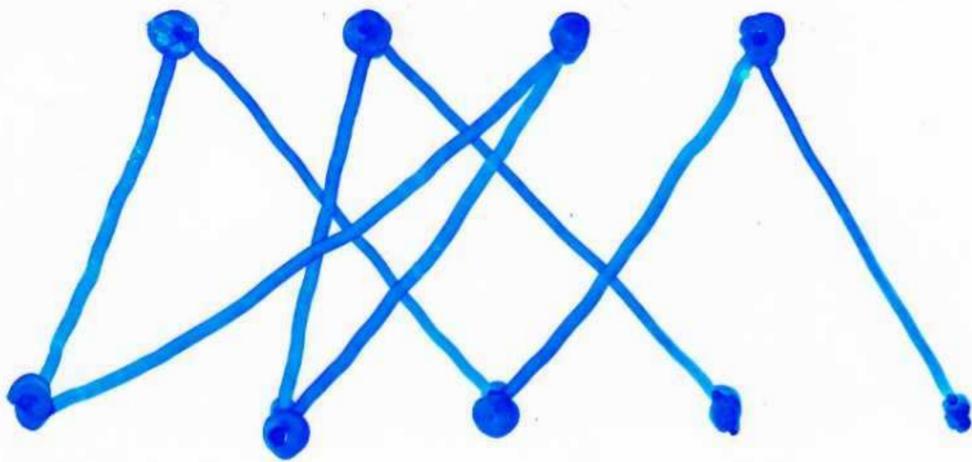
M'
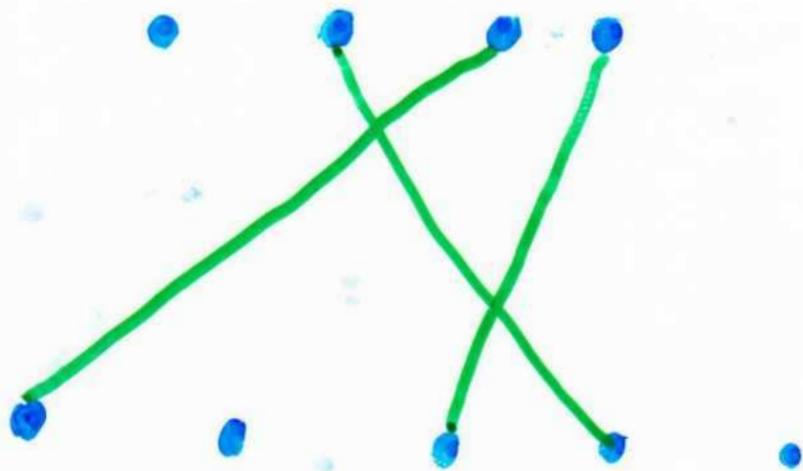
M
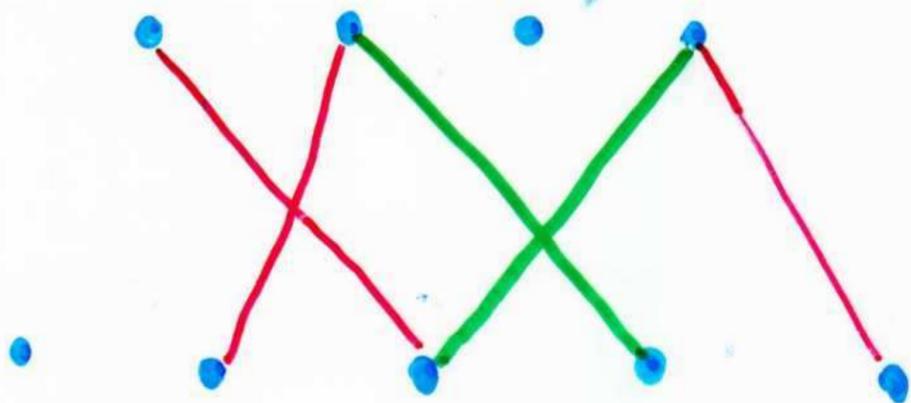
$$H = (M \cup M') - (M \cap M')$$

The connected components of H <u>must</u> be cycles & paths, since the maximum degree of H is 2. Further, the edges on each cycle or path must alternate red-green, ie. from different matchings. Since the number of red edges on any cycle = the number of green edges, for $|M'| > |M|$ there exists at least one red-green...red augmenting path!

Bipartite Matching

We live in a world where we must pair off elements of one set with elements of another:

People ⟷ Jobs

Men ⟷ Women

Tasks ⟷ equipment

If we represent each element by a vertex and draw an edge between elements of different sets which can be matched, we create a bipartite graph:

Abe   Amy

Bill   Betty

Chris   Chris

Dave   Dave
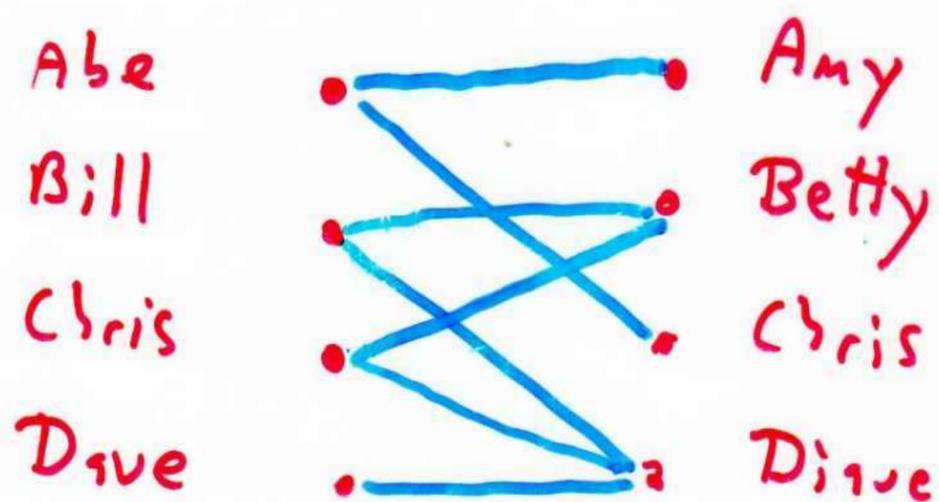
Bipartite Matching is an important special case of matching in graphs.

Given a set of boys B, and a set of girls G, is it possible for each Boy to marry a girl he knows without bigomy?

In graph theoretic terms, when does a bipartite graph have a matching of size $|B|$, where $V = B \cup G$ and $|B| \leq |G|$?

Hall's Marriage Theorem: Such a matching exists iff any subset of boys knows at least as many girls between them.

It is clear that this condition is necessary, since if $|B|$ boys know less than $|B|$ girls, each can't get a girl, without sharing.

Abe · ———— · Amy

Bill ·   · Betty

Chris ·   · Chris

Dave ·   · Diane
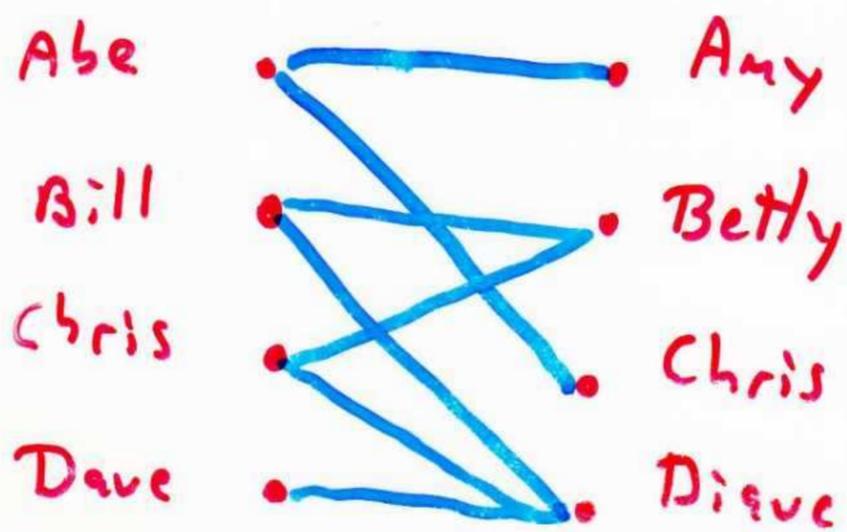
Since {Bill, Chris, Dave} only know {Betty, Diane} No matching exists.

Proof of Hall's Theorem:

By induction on the number of boys M.

For $M=1$, a matching obviously exists if the boy knows at least one girl.

Assume true for up to $M-1$ boys.

Case 1: Suppose every subset of $K < M$ boys knew at least $K+1$ girls....

Then marry off any boy to any girl he knows. This leaves a smaller problem which satisfies Hall's condition, & by induction can be matched.

Case 2: There exists a subset of $K < M$ boys which knows exactly K girls.

By induction, these K boys can be married off, leaving $N-K$ boys. Suppose now there exist a set of h boys who know less than h remaining girls (for otherwise we can complete the matching by induction)
But then these $k+h$ boys in the original graph knew less than $k+h$ girls, a contradiction!

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The Most Important Algorithm Problem you will ever solve

Given a set of numbers, find which is the largest one.

15   96   35   42   7   13   84

Ah, But what if the numbers are given to you one at a time, & you cannot go back: N=5

75
14
33
88
92

Finding the Optimal Mate

Most of us seek Mr. or Mrs. Right, the single spouse which will maximize our happiness. But how do we recognize them when we see them?

The only way to determine someone's absolute desirability is to compare them with everyone else. But we are normally able to date (evaluate) one person at a time, or else they get mad and ditch us.

Abstracting the problem — suppose we can potentially date N people in our lifetime, one at a time, no peeking ahead, no going back.

Suppose we are dating the ith person. We must decide whether to marry them before looking at the $(i+1)$st person.

What strategy can we use to maximize our chances of getting the best?

what I see | what the future holds

W_1

$3\ 5\ 1\ 4\ 2$
$W_1\ W_2\ W_3\ W_4\ W_5$

$W_1 < W_2$

$5\ 1\ 4\ 2$

$W_3 < W_1 < W_2$

$1\ 4\ 2$

$4\ 2$

Should I marry W_3 or try my luck further?

If any permutation of women is equally likely, and none of them will turn me down (a happy if unlikely assumption), what strategy will maximize my chance of getting #1?

Clearly, the probability that the ith person I see is best is $1/N$, and so it might be the first person I encounter!

The optimal strategy is to look at some
fraction of the N people, reject them all, and
pick the next person better than anyone I have
seen to date.

Suppose I reject the first $N/2$ people.
At least $1/4$ of the time I get the best

$$w_1 \quad w_2 \cdots \quad w_{N/2} \,\big|\, w_{N/2+1} \cdots\cdots \quad w_N$$

Pr (#2 is in the first half) $= 1/2$
Pr (#1 is in the second half) $= 1/2$

If these were independent (they almost are), I
win this way $1/4$ of the time! Plus there are
other ways I win :

$$\cdots 3 \cdots \big| \cdots 1 \cdots 2 \cdots$$

In fact, the optimal strategy is to wait $1/e^{th}$
of your life — so don't be fooled by the
very first one...